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MPT Versus $\mathscr{C}\mathscr{P}\mathscr{T}$: A Manifestly Covariant Presentation of Motion Reversal and Particle-Antiparticle Exchange

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We show that particle-antiparticle exchange and covariant motion reversal are two physically different aspects of the same mathematical transformation, either in the prequantal relativistic equation of motion of a charged point particle, in the general scheme of second quantization, or in the spinning wave equations of Dirac and of Petiau–Duffin–Kemmer. While, classically, charge reversal and rest mass reversal are equivalent operations, in the wave mechanical case mass reversal must be supplemented by exchange of the two adjoint equations, implying $\psi \rightleftharpoons \bar{\psi}$.

Denoting by M the rest mass reversal, P the parity reversal, T the Racah time reversal, and Z the $\psi \rightleftharpoons \bar{\psi}$ *exchange, the connection with the usual* $\mathscr{C}\mathscr{P}\mathscr{T}$ *scheme of charge conjugation, parity reversal, and Wigner motion reversal, is*

$$\mathscr{C} = MZ, \qquad \mathscr{P} \equiv P, \qquad \mathscr{T} = TZ$$

with, of course,

$$MPT = 1, \qquad \mathscr{C}\mathscr{P}\mathscr{T} = 1$$

1. INTRODUCTION

$\mathscr{C}\mathscr{P}\mathscr{T}$ invariance is a very fundamental principle in quantum field theory and elementary particle physics. The $\mathscr{C}\mathscr{P}\mathscr{T} = 1$ formula has (orthochronous) Lorentz invariance, and constitutes part of a generalized Lorentz transformation including space reversal, time reversal, and particle-antiparticle exchange.

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It thus displays a new sort of connection between geometry and physics, after those of Einstein's general relativity, and of the Einstein–de Broglie relation between momentum-energy and 4-frequency, or Heisenberg's uncertainty relations.

Although it is invariant, the $\mathscr{C}\mathscr{P}\mathscr{T}=1$ formula does not display *manifest* invariance. This is primarily because, in it, the $\mathscr{T}$ operation, Wigner's[1] "motion reversal," is not defined the same way as the $\mathscr{P}$ operation termed "parity reversal." Therefore the $\mathscr{C}$ operation, particle-antiparticle exchange, does not display at first sight explicit Lorentz invariance.

This situation somewhat resembles the one existing for momentum and energy conservation before the advent of the "manifestly covariant" quantum electrodynamics of Tomonaga, Schwinger, Feynman, and Dyson. Then, the accepted formalism displayed momentum, but not energy, conservation, in the emission or absorption of virtual particles, the rest mass of which was taken the same as for free particles. Energy conservation was restored only at the end of the calculation. By exchanging one constraint and one degree of freedom, the new formalism ensured at every step momentum-energy conservation by means of a freely adjustable rest mass—thus restoring "manifest covariance."

The present paper aims at performing something analogous with the $\mathscr{C}\mathscr{P}\mathscr{T}=1$ theorem, first by replacing Wigner's motion reversal $\mathscr{T}$ by Racah's[2] "time reversal" T, the definition of which is similar to that of the "space reversal" operation $\mathscr{P}$, which we retain, denoting it $P\equiv\mathscr{P}$. We are thus led to define an operation $M\equiv PT$ which, by inspection of the Dirac spin 1/2, or of the Petiau–Duffin–Kemmer[3] (P.D.K.) spin 1 or 0, wave equations, is seen to be rest mass reversal.[2] This was expected, as charge reversal and rest mass reversal are equivalent in the classical relativistic equation of motion of a charged particle—which remark underlines the Stueckelberg–Feynman interpretation of antiparticles.

However, inspection of the Dirac or the P.D.K.[3] equation shows that the $e\rightleftharpoons -e$ or $m\rightleftharpoons -m$ equivalence requires there exchange of the equation with its adjoint equation. This we denote the Z, or $\psi\rightleftharpoons\bar{\psi}$, exchange. Then we show that MZ is none else (up to a unitary transformation devoid of physical content) than the (relativistically invariant) *particle-antiparticle exchange* operation $\mathscr{C}$ of the existing formalism.

Coming back to the $PT\equiv M$ operation, which, in this form, obviously has to do with space and time reversal, we first remark that *it is not exactly*

[2] Tiomno[4] and Sakurai[5] have discussed rest-mass reversal in a different context.

that. The very obvious reason is that it reverses, in the wave equation, the sign of the scalar product $\gamma^i p_i$, or $\beta^i p_i$, of the 4-velocity operator by the momentum-energy operator.[3] But then, as charge reversal in the classical relativistic equation, is equivalent to proper time reversal, (which reverses the 4-velocity but not the 4-acceleration), this can be interpreted as *covariant motion reversal*, bringing the particle backwards through its previous x, y, z, *and* t coordinates—very much like a movie film run backwards brings back, in reversed order, the space coordinates *and* the time coordinate, as displayed on the clocks pictured in the film. Then, the mathematical reasoning must go exactly like before, and we see that, for describing the covariant motion reversal, we must associate the Z exchange $\psi \rightleftharpoons \bar{\psi}$ with the $MT \equiv P$ operation. But, now, $\psi \rightleftharpoons \bar{\psi}$ means *emission-absorption exchange* (or *preparation-measurement exchange*).

So, the $MZ \equiv PTZ$ operation can be interpreted either, passively, as particle-antiparticle exchange, or, actively, as covariant motion reversal. And this, being in full conformity with both the classical invariances of the relativistic equation of motion of a charged particle, and of the axiomatics of second quantization, is also in deep harmony with Feynman's theory of the S matrix.

As for the connection with the existing $\mathscr{C}\mathscr{P}\mathscr{T} = 1$ scheme, we have already noted that $MZ \simeq \mathscr{C}$, this being a relativistically invariant definition. Thus $PTZ \simeq \mathscr{P}\mathscr{T}$. And, as we retain the parity operation, $P \equiv \mathscr{P}$, it turns out that TZ is none else than the Wigner motion reversal: $TZ \simeq \mathscr{T}$.

In the extreme relativistic, or zero rest mass, limit, of the spin 1/2 wave equation, we fall back exactly upon the existing 2-component Weyl theory of paired "screwons" (one of which is physically excluded). In this limit an exact compensation $TZ \simeq \mathscr{T} = 1$ occurs—and occurs invariantly. Thus, $\mathscr{C}\mathscr{P} = 1$, with $\mathscr{C} = \mathscr{P} \neq 1$. This holds in all cases where parity violation is due to neutrinos.

On the whole, our *manifestly covariant* $MPT = 1$ scheme is fully consistent with the existing $\mathscr{C}\mathscr{P}\mathscr{T} = 1$ scheme—as it had to be. But it disentangles the underlying mathematics, by emphasizing the role of the $\psi \rightleftharpoons \bar{\psi}$ exchange, with its two physically different (through mathematically equivalent) interpretations. It thus displays in full light the *equivalence between particle-antiparticle exchange and covariant motion reversal.*

As for the above identity operation 1, it is none else than *true geometrical reversal of all four spacetime axes.*

[3] A familiar remark is that, while all four γ's are Hermitean, the three p's are Hermitean and p^4 anti-Hermitean. As is well-known, this is taken care of by the appropriate definition $\bar{\psi} \equiv \psi^+ \gamma^4$.

2. CLASSICAL PREAMBLE

The relativistic equation of motion of a charged point particle of rest mass m and charge e inside an electromagnetic field $H^{ij}(i, j, k, l = 1, 2, 3, 4; c = 1; x^4 = it; u_i u^i = -1; \dot{u}^i = du^i/d\tau)$ reads

$$m\dot{u}^i = eH^{ij}u_j \tag{1}$$

it is invariant under the substitution $e \rightleftharpoons -e$ and $m \rightleftharpoons -m$, and this is the basis of the Stueckelberg–Feynman theory of particle-antiparticle exchange, as $m \rightleftharpoons -m$ entails $mu^i \rightleftharpoons -mu^i$.

Instead of reversing e or m, internal properties of the particle, we could just as well reverse the external field H^{ij}. All three operations reverse the relative sign of the 4-velocity and the 4-acceleration. But this can also be done by reversing the sign of the proper time τ along the trajectory, yielding $u^i \to -u^i$ and $\dot{u}^i \to +\dot{u}^i$. The latter two operations are different aspects of covariant motion reversal, and are related as follows:

Theorem. The space time symmetry $x'^i = -x^i$ together with $H^{-ij}(x^{-l}) = H^{ij}(-x^l)$[4] is the identity operation.

As for covariant motion reversal *via* $\tau' = -\tau$, it brings the particle backwards along the same trajectory through its "previous" *space* $x, y, z,$ *and time* t coordinates, very much like running backwards a movie film, as said in Section 1.

Finally, *in classical relativistic mechanics, particle-antiparticle exchange and covariant motion reversal are mathematically equivalent*—a fundamental trait holding also by *correspondence* in relativistic quantum mechanics, as we will see.

3. SECOND QUANTIZATION: MOTION REVERSAL AND PARTICLE-ANTIPARTICLE EXCHANGE AS THE ACTIVE AND PASSIVE ASPECTS OF THE SAME OPERATION

Axiom. Emission of a particle and absorption of an antiparticle are mathematically identical operations.

This axiom is basic in the second quantization theory—including the Dirac and the Stueckelberg–Feynman interpretations of negative energy particles as antiparticles.

It is tacitly assumed that the dynamical state is the same for the emitted particle or the absorbed antiparticle: same momentum-energy etc. If there is

[4] This active spacetime symmetry obviously exchanges retarded and advances waves.

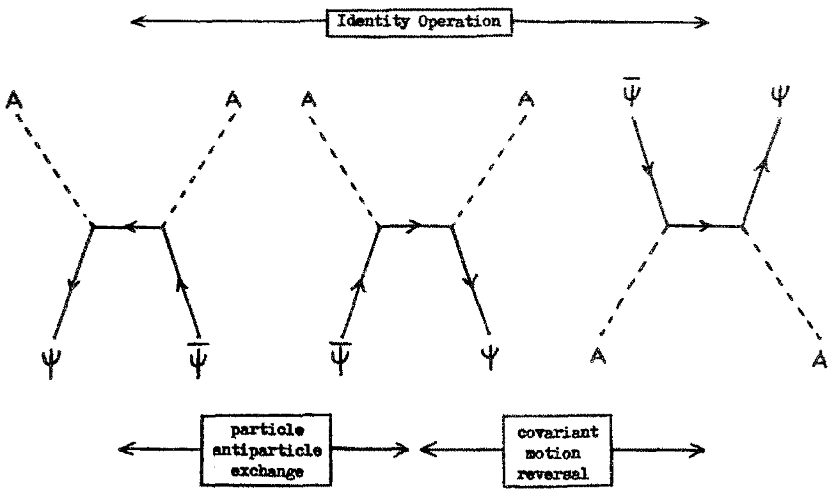

Fig. 1. Particle-antiparticle exchange and covariant motion reversal as the passive and active aspects of the same operation: true reversal of the spacetime axes.

some ambiguity in the "etc...", the above axiom may help in clarifying it. An important example will be given below.

Theorem. Exchange of particle and antiparticle (at emission or absorption), and exchange of emission and absorption of a particle (in a given state), are mathematically identical operations. This follows directly from the axiom.

Definition. Exchange of emission and absorption of a particle (in a given state) is motion reversal.

Corollary. Particle-antiparticle exchange and motion reversal are mathematically identical operations.

In fact, they are the passive and the active aspect of the same operation, as exemplified in Fig. 1 (*via* the example of the annihilation (or creation) of an e^+e^- pair into (or from) a photon pair, both pairs being E.P.R. correlated[5]).

In the presently current jargon, motion reversal is (noncovariantly) represented by Wigner's $\mathscr{T}$ reversal, so that, *via* $\mathscr{C}\mathscr{P}\mathscr{T}$ invariance, what has

[5] For a relativistically covariant discussion of the E.P.R. correlation of either two measurements issuing form a common preparation, or of two preparations converging into a common measurement, see O. Costa de Beauregard.[6]

just been called particle-antiparticle exchange is represented (noncovariantly) by $\mathscr{C}\mathscr{P}$. When Lee and Yang[7] demonstrated, in 1956–57, that $\mathscr{C}\mathscr{P}$ is a more general symmetry than $\mathscr{C}$ or $\mathscr{P}$, this came as a surprise. Here, we have found that it is quite natural to define particle-antiparticle exchange as mathematically identical to motion reversal.

However, considering the Feynman graphs of Fig. 1, it should be clear that there is no reason why one should not, and every reason why one should, define both particle-antiparticle exchange and motion reversal as a relativistically invariant operation (which, as we have seen in Section 1, was the case in the prequantal relativistic scheme).

Therefore, our problem is now to define accordingly a new *MPT* scheme equivalent to the existing $\mathscr{C}\mathscr{P}\mathscr{T}$ scheme, the guess being that it will *correspond* to the prequantal relativistic scheme discussed in Section 2.

4. COVARIANT MOTION REVERSAL AND PARTICLE-ANTIPARTICLE EXCHANGE IN THE DIRAC SPIN-1/2 EQUATION

It is well-known (and evident, as $\gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4$ anticommutes with all four γ's) that the Dirac equation (in units such that $c = 1$ and $\hbar = 1$)

$$\{\gamma^i(i\partial_i - eA_i) + m\}\,\psi = 0 \tag{2}$$

is invariant under the exchange $\psi \rightleftarrows \gamma^5\psi$ and $m \rightleftarrows -m$. This operation we denote M.

However, in Eq. (2), this is not equivalent to changing the sign of e. But this can be amended by considering also the adjoint equation, where $-i\underset{\leftarrow}{\partial}_i$ replaces $i\underset{\rightarrow}{\partial}_i$. In transposed form it reads

$$\{\tilde{\gamma}^i(i\partial_i + eA_i) + m\}\,\tilde{\bar{\psi}}' = 0 \tag{3}$$

with

$$\psi' = \gamma^5\psi \tag{4}$$

As the $\tilde{\gamma}^i$'s are a set of Dirac matrices, there exists (according to a well-known Pauli theorem) a unitary transformation such that

$$C\tilde{\gamma}^i C^{-1} = \gamma^i, \qquad C\tilde{\bar{\psi}}' = \tilde{\bar{\psi}}'' \tag{5}$$

but this is merely a representation change, devoid of physical content, because, for all sixteen γ's[6]

$$C\tilde{\gamma}C^{-1} = \gamma \tag{6}$$

Therefore, denoting Z or, symbolically, $\psi \rightleftharpoons \bar{\psi}$, the exchange of the adjoint Eqs. (2) and (3), we see that the MZ operation is none else than particle-antiparticle exchange:

$$\mathscr{C} \simeq MZ \tag{7}$$

we have set the symbol $\simeq$ rather than $=$ because of the (not compulsory) change in representation (5). So, *combined with* $\psi \rightleftharpoons \bar{\psi}$, *rest mass reversal does represent particle-antiparticle exchange*—in close *correspondence* with the classical state of affairs discussed in Section 2.

Of course, the M operation can also be understood as reversing all four signs in the scalar product $\gamma^i(i\partial_i - eA_i)$. It is generally stated in the literature that this is reversal of all four axes, or Racah space-time reversal. This, however, is a questionable denomination, as a scalar product should not change sign under reversal of any axis. Similar remarks hold, of course, for the so-called "space reversal" or "parity reversal" P, defined as multiplication from the left of equation (2) by γ^4 and substitution $\psi' = \gamma^4\psi$, and for the "Racah time reversal" T, multiplication from the left by $\gamma^1\gamma^2\gamma^3$ and substitution $\psi' = \gamma^1\gamma^2\gamma^3\psi$. Also, we remark that the Dirac 4-current $i\bar{\psi}\gamma_i\psi$ and the Gordon 4-current $i\bar{\psi}[\partial_i]\psi - eA_i\bar{\psi}\psi([\partial_i] \equiv \frac{1}{2}[\underset{\rightarrow}{\partial_i} - \underset{\leftarrow}{\partial_i}])$ transform differently under P, T, and PT. And also, that the Tetrode momentum-energy density $i\bar{\psi}[\partial_i]\gamma_j\psi - eA_i\bar{\psi}\gamma_j\psi$, being a second rank tensor, should not change sign under the PT operation.

So, what is the true meaning of the PT operation? As $\gamma^i(i\partial_i - eA_i)$ is the scalar product of the 4-velocity operator by the momentum-energy operator,[2] the guess is that it must be covariant motion reversal. Indeed, we have seen in Section 2 that the latter can be classically represented as reversal of the external field, which is, here, A_i. And, of course, in Eq. (2), this is equivalent to changing the sign of e.

We can then repeat exactly the argument used for rest mass reversal, and show that *$PTZ \equiv MZ$ does represent covariant motion reversal.* However, the physical interpretation of the $\psi \rightleftharpoons \bar{\psi}$ exchange has changed: here, it means *emission-absorption exchange*, or *preparation-measurement exchange.*

This is a synthesis of the contents of Sections 2 and 3, and is fully consistent with Feynman's theory of the S matrix.

[6] If one proceeds in one step, bypassing the mass reversal, *via* $\gamma^i = -C\tilde{\gamma}^iC^{-1}$, then the physics is in the unitary transformation, as $\gamma = \pm C\tilde{\gamma}C^{-1}$.

As for the connection with the existing $\mathscr{C}\mathscr{P}\mathscr{E}$ scheme,

$$\mathscr{P} \equiv P \qquad \text{and} \qquad \mathscr{E} \simeq TZ \tag{8}$$

are the (noncovariantly defined) "parity reversal" and "Wigner motion reversal." Of course,

$$MPT = \mathscr{C}\mathscr{P}\mathscr{E} = 1 \tag{9}$$

Incidentally, $\mathscr{C}$ invariance of quantum electrodynamics needs (as is well-known) that $\mathscr{C}$ noninvariance of Dirac's equation be corrected by use of the current

$$j^i = \frac{e}{2}\,[\bar{\psi}\gamma^i\psi - \tilde{\psi}'\tilde{\gamma}^i\tilde{\bar{\psi}}']$$

5. THE NEUTRINO-ANTINEUTRINO PAIR OF "SCREWONS"

In the extreme relativistic, or zero rest mass limit, we retain the previous PTZ definition of particle-antiparticle exchange or covariant motion reversal.

In that limit, the $\gamma^5\psi$ transformed from a solution ψ of the Dirac equation also is a solution, so that the ψ and $\gamma^5\psi$ pair can be expressed as superpositions of the two orthogonally projected solutions

$$\varphi \equiv R\psi = -R\gamma^5\psi, \qquad \chi \equiv L\psi = L\gamma^5\psi \tag{10}$$

where

$$R \equiv (1-\gamma^5)/\sqrt{2}, \qquad L \equiv (1+\gamma^5)/\sqrt{2} \tag{11}$$

so that

$$RL = LR = 0, \qquad R^2 = R, \qquad L^2 = L \tag{12}$$

We notice that

$$\bar{\varphi} = \bar{\psi}L = \bar{\psi}\gamma^5 L, \qquad \bar{\chi} = \bar{\psi}R = -\bar{\psi}\gamma^5 R \tag{13}$$

and

$$\bar{\varphi}\varphi = \bar{\chi}\chi = \bar{\varphi}\gamma^5\psi = \bar{\chi}\gamma^5\chi = 0 \tag{14}$$

The operation PT transforms φ and χ into themselves:

$$PT\varphi \equiv \gamma^5\varphi = -\varphi, \qquad PT\chi \equiv \gamma^5\chi = +\chi \tag{15}$$

while P or T exchanges them:

$$P\varphi' = T\varphi'' = \chi, \qquad P\chi' = T\chi'' = \varphi \tag{16}$$

Choosing a representation where γ^5 is diagonal simplifies matters, as φ and χ then have only two nonzero components, and the Dirac equations break down into two pairs of two equations. As then $\tilde{\gamma}^5 = \gamma^5$, from (10) and (13) we get

$$Z\bar{\varphi} = \chi, \qquad Z\bar{\chi} = \varphi \tag{17}$$

so that (remembering that $\mathscr{C} = PTZ$)

$$Z = P = T = \mathscr{C} \tag{18}$$

In 1956–1957 there came a big surprise. Lee and Yang[(7)] announced, and experimentation verified, that the physical neutrino always has positive, and the antineutrino negative, helicity. This allows to universally exclude one of the two pure helicity solutions, say χ, and to decide that φ represents either a positive energy-positive helicity neutrino, or a negative energy-negative helicity antineutrino, (each corresponding to one of the two components of the φ). This is perfectly consonant with the Stueckelberg–Feynman definition of antiparticles.

It goes without saying that this chirality property of Nature takes us beyond the realm of the tensorial calculus *stricto sensu*: when excluding χ, the associated definitions (10) and (13) are Lorentz and PT invariant, but neither P nor T invariant.

As for the deduction of the pair of associated Weyl "screwon" equations in a representation where γ^5 is diagonal, we refer to the literature.[7]

6. COVARIANT MOTION REVERSAL AND PARTICLE-ANTIPARTICLE EXCHANGE IN THE PETIAU–DUFFIN–KEMMER THEORY OF SPIN 1 OR 0 PARTICLES

It so happens that, although the algebra of the D.D.K.[(3)] β matrices differs greatly from that of the Dirac γ's, statements closely paralleling those of Section 4 still hold.

Here, the "space reversal" P is obtained *via* multiplication of the P.D.K. equation ($i = 1, 2, 3, 4$)

$$\{\beta^i(i\partial_i - eA_i) + m\}\psi = 0 \tag{19}$$

[7] See for example J. Hamilton,[(8)] p. 138–139.

from the left by η_4, belonging to the set of

$$\eta_i \equiv 2\beta_i^2 - 1 \tag{20}$$

which are diagonal with eigenvalues $+1$ and -1, and such that

$$\eta_i\beta_j + \eta_j\beta_i = 0 \qquad \textit{if} \quad i \neq j \quad \textit{and} \quad \eta_i\beta_i = \beta_i\eta_i \tag{21}$$

The "spacetime Racah reversal" is obtained *via* multiplication by

$$\eta_5 \equiv \eta_1\eta_2\eta_3\eta_4 \tag{22}$$

and substitution $\psi' = \gamma_5\psi$, which reverses the sign of m. So, we denote $M = PT$ this operation.

All other items remain similar to those of Section 4, *modulo* the substitutions $\gamma^i \to \beta^i$ in the wave equations and $\gamma_i \to \eta_i$ for "reversing" axes j, k, l.

The Z, or $\psi \rightleftharpoons \bar{\psi}$ operation, is defined as before, but, as the β's are self transposed, the unitary transformation (5) is unnecessary.

An additional remark is that the η's being diagonal with eigenvalues $+1$ and -1, each element of $\eta_i\psi$ is equal to the corresponding one in ψ up to a sign. So, for being able to handle both the "proper" and "pseudo" scalar, vector, or tensor cases, both options $\psi' = \pm\eta_i\psi$ should be retained.

10. CONCLUSIONS

We have shown that particle-antiparticle exchange and covariant motion reversal are two physically different aspects of the same relativistically invariant mathematical operation usually denoted as $\mathscr{C}$. In relation to the spinning wave equations of Dirac or Petiau–Duffin–Kemmer, this operation is equivalent to the product of rest mass reversal by exchange or the two adjoint equations, so we denote it MZ, where, symbolically, $\psi \rightleftharpoons \bar{\psi}$ stands for Z. Thus $\mathscr{C} = MZ$.

Evidently, the M operation is equivalent to PT, the product of "parity reversal" by "Racah time reversal," so that $\mathscr{C} = MZ \equiv PTZ = \mathscr{P}\mathscr{T}$, where $\mathscr{T}$ stands for the "Wigner motion reversal," and one has $\mathscr{P} \equiv P$ and $\mathscr{T} = TZ$. In other words, covariant motion reversal PTZ is usually expressed as the product of (noncovariant) parity reversal by Wigner motion reversal.

In the extreme relativistic, or zero rest mass limit of the spin 1/2 case, this scheme is strictly equivalent to the Weyl theory of twin two component 'screwons" of opposite helicities. It so happens that, in this case, the exact compensation $TZ = \mathscr{T} = 1$ occurs invariantly, and that the operation $\mathscr{C} = \mathscr{P}$ exchanges the "screwons."

We have also shown that this scheme strictly *corresponds* to the equivalence of the $e \rightleftarrows -e$ or $m \rightleftarrows -m$ exchanges in the prequantal relativistic equation of motion of a charged particle, and strictly adheres to the basic principles of second quantization. Therefore it is in deep harmony with the Feynman S-matrix theory.

Finally, the identity operation in $MPT = \mathscr{C}\mathscr{P}\mathscr{T} = 1$ is none else than the true geometrical reversal of all four spacetime axes (see Fig. 1).

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