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Relativistic Dynamics of Interacting Point Particles: Central Position of the Wheeler–Feynman Scheme

O. Costa de Beauregard[1]

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The Wheeler–Feynman (WF) relativistic theory of interacting point particles, generalized by acceptance of an arbitrary spacelike interaction, is shown to possess a privileged status, reminiscent of the "central force" interactions occurring in Newtonian mechanics. This scheme is shown to be isomorphic to the classical one of the statics of interacting flexible current-carrying wires obeying the Ampère–Laplace (AL) formulas: to the tension $\mathbf{T}$ ($\mathbf{T}^2 = const$) *of the wire corresponds the momentum-energy* p^i ($p_i p^i = -c^2m^2$) *of the particle; to the Laplace linear force density* $-i\mathbf{H}\times \mathbf{dr}$ *corresponds the Lorentz force* $QH^{ij}\,dr_j$*; to the Laplace potential* $ir^{-1}\,\mathbf{dr}$ *corresponds the WF potential* $Q\delta(r^2)\,dr^i$*, etc. Among the differences, there is self-action in the AL scheme and no self-action in the WF scheme. A stationary energy principle in the AL scheme is isomorphic to Fokker's stationary action principle in the WF scheme.*

1. INTRODUCTION

In the Newtonian mechanics of interacting point particles the case of central forces, with an arbitrary distance law, was preferred paradigm, where the formalization was very straightforward, easily yielding "general theorems." This paper aims at showing that, in the special-relativistic dynamics of interacting point particles, the elegant Wheeler–Feynman[(1)] scheme, generalized so as to accommodate an arbitrary spacelike interaction,[(2)] is favored in an analogous way. It goes without saying that a spacelike interaction, with any pair of line elements on the space-time trajectories interacting with each other, is the most straightforward generalization of Newton's direct interaction at a distance.

[1] Institut Henri Poincaré, 11 Rue P. et M. Curie, 75005, Paris.

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At this point the isomorphism between the classical statics of filaments and the relativistic dynamics of point particles that I have stressed elsewhere[3] should be recalled. The correspondence table ($x^4 = ict$; $i, j, k, l = 1, 2, 3, 4$; $V^i \equiv dx^i/d\theta$, $V_i V^i = -c^2$) is shown in Table I.

This table can be generalized[3] so as to accommodate "angular tensions" and angular momenta (Weyssenhof's equations[4]), but this is not needed here.

It is clear that, developed along such lines, the relativistic dynamics of interacting point particles should display features corresponding to those of, say, interacting current-carrying wires, each element of a wire interacting with each element of another wire. What then comes out goes beyond what was expected: while (as is well known) the Lorentz force $F^i \equiv QH^{ij}V_j$ applied to a point charge Q by a magnetoelectric field H^{ij} exactly corresponds to the Laplace linear force density $-i\mathbf{H} \times \mathbf{t}$ applied to a current element $i\mathbf{t}$, it turns out that the Wheeler–Feynman (WF) field strength generated by a line element also corresponds exactly to the Laplacian one—and so do the associated 4-potential A^i and vector potential $\mathbf{A}$. The correspondence table is (φ' denoting the derivative of φ with respect to its argument) shown in Table II.

However, there are differences between the two cases. In the WF case, as the trajectories are timelike and the interaction is lightlike (or spacelike[2]), there is no self-action. In the Ampère–Laplace (AL) case of current-carrying wires no such restriction occurs, so there is a self-action, each element of one wire interacting with each element of the same wire. Incidentally, in Dettman and Schild's[5] generalization of the Wheeler–Feynman scheme, timelike interaction is not excluded, so there is self-action.

Table I. Correspondence Table

Euclidean 3-space $\mathscr{E}$	Minkowskian 4-space $\mathscr{M}$
Filament $r(\theta)$	Timelike trajectory $r^i(\theta)$
Curvilinear abscissa θ	Proper time θ
Line element $\mathbf{dr}$	Line element dr^i
Unit tangent vector $\mathbf{t} \equiv \mathbf{dr}/d\theta$	4-velocity $V^i \equiv dr^i/d\theta$
Linear force density $\mathbf{f}$	Applied force F^{ij} or $F^i \equiv F^{ij}V_j$
Tension $\mathbf{T}$	Momentum-energy p^i
Equation $\mathbf{dT} = \mathbf{f}\, d\theta$	Equation $dp^i = F^{ij}\, dr_j = F^i\, d\theta$
Flexible filament $\mathbf{t} \times \mathbf{T} = \mathbf{0}$	Spinless particle $V^i p^j - V^j p^i = 0$
Special case $\mathbf{t} \cdot \mathbf{f} = 0$, $T^2 = \text{const}$	Special case $F^{ij} = -F^{ji}$, $F^i V_i = 0$ $p^i p_i - c^2 m^2 = \text{const}$

Table II. Correspondence Table

Current intensity i	Electric charge Q
Laplace field	WF field
$\mathbf{dH} = -i\varphi'(r^2)\mathbf{r}\times\mathbf{db}$	$dH^{ij} = Q\varphi'(r^2)[r^i\,db^j - r^j\,db^i]$
Laplace potential	WF potential
$\mathbf{dA} = i\varphi(r^2)\,\mathbf{db}$	$dA^i = Q\varphi(r^2)\,db^i$

Another difference is that, in the WF case, a conservation theorem holds for the total momentum-energy, with no analog in the Ampère–Laplace case.

Finally, the whole array of formulas is deducible from an extremum principle: Fokker's stationary action principle in the WF case, and a corresponding stationary energy principle in the AL case.

The variation must be carried out with the charges Q or the intensities i (respectively) kept constant—a clear mathematical prescription, but not an easily realizable one in the AL case.

2. A PRIVILEGED EXPRESSION OF ACTION AND REACTION: THE WHEELER–FEYNMAN BRACKET

As mentioned in the Introduction, we require that the distance dependence of the interaction between two line elements da^i and db^i (which we denote by $\varphi'(r^2_{ab})$ because it will go into $\varphi(r^2_{ab})$ when using the potentials) be zero when r^i_{ab} is timelike.

The magnitudes entering the expression we are looking for are the three 4-vectors da^i, db^i, r^i_{ab}, the function φ' or φ, and the two charges Q_a and Q_b. As the length of the momentum-energy $p^i \equiv mV^i$, $p^ip_i = -c^2m^2$, must be preserved, the force $Q_aH^{ij}_{(a)}$ created at a^i by b^i must be skew symmetric and thus propertional to $r^i_{ab}\,db^j - r^j_{ab}\,db^i$, so that

$$d^2p^i_a = Q_aQ_b\varphi'(r^2_{ab})[r^i_{ab}\,db^j\,da_j - r^j_{ab}\,db^i\,da_j]$$

dp^i_a is obtained by the integration $\int_{-\infty}^{+\infty} db$ of this expression. Let it be remarked that, by the substitution $\mathscr{M} \to \mathscr{E}$, $p^i \to \mathbf{T}$ and $Q \to i$, this formula becomes identical to the Laplace formula for currents, provided that the wire is perfectly flexible (i.e., not resisting torque).

If this formula is to yield an action-and-reaction statement, the bracket must be completed by the term $-r^j_{ab}\,da^i\,db_j$, and this term must vanish in the integration $\int_{-\infty}^{+\infty} db$. This is indeed the case, as $2r^j_{ab}\, db_j = d_b[(r^j_{ab})^2]$, $\varphi'(r^2)\, dr^2 \equiv \varphi(r^2)$, and

$$\varphi(r^2) = 0 \text{ if } r^2 \text{ is timelike} \tag{1}$$

Finally

$$d^2p^i_a = -d^2p^i_b = Q_a Q_b \varphi'(r^2_{ab})[r^i_{ab}\, da^j\, db_j - r^j_{ab}(db^i\, da_j + da^i\, db_j)] \tag{2}$$

The bracket in formula (2) we call the WF bracket. Strictly corresponding to it is an AL bracket, also entailing an action-and-reaction statement; then the nullity of the integral $\int db$ stems from the fact that it is a closed integral, $\oint db$.

Now we remark that $-Q_b\varphi' r^j\, da_j\, db^i \equiv \frac{1}{2} Q_b d_a \varphi\, db^i$ is none other than the variation at a of the potential $A^i_{(a)}$ generated by the line element $Q_b\, db^i$. Therefore, setting

$$A^i_{(a)} \equiv Q_b \int_{-\infty}^{+\infty} \varphi(r^2_{ab})\, db^i \tag{3}$$

(whence follows $\partial_i A^i_{(a)} = 0$) and defining the "combined momentum energy" as

$$P^i_a = p^i_a + Q_a A^i_{(a)} \tag{4}$$

we rewrite formula (2) in the simplified form

$$d^2P^i_a = -d^2P^i_b = Q_a Q_b \varphi'(r^2_{ab})(r^i_{ab}\, da^j\, db_j) \tag{5}$$

we call the last parenthesis the "WF parenthesis." Let it be remarked that the right-hand side of formula (5) is of a "central force" type. An exactly corresponding argument holds in the LA formalism, the formula for $\mathbf{A}_{(a)}$ being well known.

Denoting α and β the proper times of particles a and b and applying the operator

$$\int_{\alpha}^{+\infty} \int_{-\infty}^{\beta} - \int_{-\infty}^{\alpha} \int_{\beta}^{+\infty} \tag{6}$$

to the WF bracket or parenthesis, respectively, with a spacelike separation of a and b, Wheeler and Feynman obtain expressions for the "potential momentum-energy" p^i_{ab} or the "combined potential momentum-energy" P^i_{ab}, respectively, and the conservation theorem

$$p^i_a + p^i_b + p^i_{ab} = P^i_a + P^i_b + P^i_{ab} = \text{const} \tag{7}$$

(without correspondence in the LA formalism).

Of course the whole argumentation, here presented for two interacting charged point particules or current elements, is easily extended to the case of n particles or current elements interacting two by two.

The whole set of preceding formulas is easily derived[1,3] from an extremum principle: the Fokker[6] action principle $\delta\mathscr{A}=0$ applied to the expression

$$\mathscr{A} \equiv -\sum_a \int p_a^i \, da_i + \sum_{a \neq b} Q_a Q_b \iint \varphi(r^2) \, da^i \, db_i \tag{8}$$

in the WF scheme, the energy principle $\delta\mathscr{W}=0$ applied to the expression

$$\mathscr{W} \equiv -\sum_a \oint \mathbf{T}_a \cdot \mathbf{da} + \sum_{a \& b} i_a i_b \iint \varphi(r^2) \, \mathbf{da} \cdot \mathbf{db} \tag{9}$$

in the AL scheme.

Finally, restriction to the electromagnetic case is obtained as shown in Table III.

Remark. In formula (9), with $\varphi(r^2)=1/r$, the double integral comprises the well-known mutual and self-energies of the currents.

In the simple integral, "minus" the tension T of a filament shows up as the linear density of a potential energy. Below we give two examples showing that this is a physically sound interpretation.

1. Consider a rope hanging from one extremity in a constant gravity field $\mathbf{g}$; fixing to the rope a weight mg successively at heights differing by l, we obtain two configurations where the potential energy differs by $mgl \equiv Tl$. The location of this potential energy must be in the rope. Therefore $-T$ is the (negative) linear density of potential energy contained in a stressed filament.

2. Consider two pulleys of equal radius R connected by a belt and rotating with an angular velocity Ω. One pulley acts as a motor and the other as a receiver, so that torques $+C$ and $-C$ are respectively applied to them and the tensions of the two straight segments of the belt are T_0+T

Table III. Electromagnetic Case

AL statics of currents	WF dynamics of charges
$\varphi(r^2)=1/r$	$\varphi(r^2)=\delta(r^2)$

and $T_0 - T$, with $C = 2RT$. If $V = R\Omega$ denotes the linear velocity of the belt, the power flowing from motor to receiver is $P = C\Omega = 2TV$. So, this power is flowing along both segments at the velocity V, with linear densities $-T$ along the stressed, and $+T$ along the unstressed, segment.

3. BRIEF CONCLUSION

Among the various schemes of relativistic dynamics of interacting point particles that have been proposed,[7–9] the one[1,2,4] that (as explained here) is isomorphic to the classical statics of interacting flexible filaments has the merits of manifest covariance, conceptual simplicity, physical plausibility, and avoidance of such unobservables as a "supertime."

Together with their generalization of the Wheeler–Feynman[1] scheme, Dettman and Schild[4] have produced the six-component angular momentum conservation theorem, entailing of course[10] the rectilinear motion law of the barycenter. Katz[2] has produced an interaction model differing from Wheeler and Feynman's electromagnetic one—these two being, up to now, the only known examples in this category.

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