Foundations of Physics, Vol. 16, No. 11, 1986

On Carmeli's Exotic Use of the Lorentz Transformation and on the Velocity Composition Approach to Special Relativity

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Received August 30, 1985; revised March 20, 1986

As shown by Ramarkrishnan, the faithful mapping, in the sense of Lie groups, of the real line onto the finite segment $-1 < u < +1$ is $u = \tanh A$, from which follows the "relativistic velocity composition law" $w = (u+v)/(1+uv)$ and the Lorentz–Poincaré transformation formulas. Composition of translations is merely one application of this. Carmeli has shown that composition of rotations is another one. There may be still others.

1. INTRODUCTION

M. Carmeli[(1)(2)] has recently used the Lorentz–Poincaré group for composing rotations instead of translations: see his formulas (10) and (46). At first sight this looks quite surprising, but much less at second sight. A theorem by Ramakrishnan,[(3)] which he used for deriving the relativistic velocity composition law, and then the Lorentz–Poincaré formulas, states that the faithful mapping, in the sense of Lie groups, of the real line $-\infty < A < +\infty$ onto a finite segment $-1 < u < +1$ is given by $u = \tanh A$. Section 2 below presents a compact derivation of Ramakrishnan's result.

As in the history of Alpine climbing, Ramakrishnan's 1973 straight way up to the Lorentz formulas via the velocity composition law has come after a long succession of preliminary explorations. It all started in 1818, when Fresnel formalized Arago's null result in an ether-wind experiment by his "ether-drag formula." Section 3 outlines very briefly this exciting story.

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Yilmaz[4] once asked "could Bradley have derived the Lorentz formulas from his aberration formula," the answer being "yes," under the assumption of invariance of the Huygens orthogonality between light rays and wave fronts, that is, also invariance of Fermat's extremum principle. Via the geometrical theory of envelopes, the Huygens–Fermat principle entails isotropy of the speed of light—assumed "invariant" by Einstein in 1905.

In a similar vein, I ask, in Section 3, "could Fresnel have derived the special relativity theory from his ether-drag formula?," the answer being "yes."

Why is it then that no "lynx-eyed" theoretician of the Fermat–Huygens–Fresnel sort arrived at the special relativity theory before Michelson's experiment? The mathematics of the day would have been quite sufficient. The blocking was in the metaphysics. The very ideas that a speed (be it that of light) should be invariantly isotropic, or that the composition law of velocities should not be addition, were then far too frightful to be stared straight at—even by a "lynx-eyed" theoretician. In those days nobody would have even thought of questioning the accepted kinematics, resting on the absolute (Euclidean) space and time concepts. And so Fresnel's, as later Fitzegerald's and Lorentz's, reasoning was expressed in dynamical, not kinematical, terms.

2. RAMAKRISHNAN'S ARGUMENT IN A NUTSHELL

Suppose we know that some physical extensive magnitude u has an upper and a lower bound and that it should be formalized as a mapping, in the sense of Lie groups, of the real axis $-\infty < A < +\infty$. By choosing appropriately the zero point and the scale, the case can be stated as $-1 < u < +1$. Then

$$u = \tanh A \tag{1}$$

is an obvious candidate.[2] Since by setting $A + B = -C$ and $(A, B, C) \rightleftharpoons (u, v, w)$ the composition law

$$\tanh(A + B) = \frac{\tanh A + \tanh B}{1 + \tanh A \cdot \tanh B} \tag{2}$$

can be cast into the form

$$\begin{aligned} A + B + C &= 0 \\ u + v + w + uvw &= 0 \end{aligned} \tag{3}$$

[2] Appendix 1 presents a derivation of this guess.

a group structure is displayed, together with its "neutral element" and "inverse transformation."

Formula (2) is identical with the "relativistic velocity composition law," and it is quite easy to proceed from it to the Lorentz formulas. For "timelike vectors" we write $u = x/ct$ and derive the Lorentz formulas as

$$\begin{aligned} \sinh(A+B) &= \sinh A \cosh B + \cosh A \sinh B \\ \cosh(A+B) &= \cosh A \cosh B + \sinh A \sinh B \end{aligned} \tag{4}$$

As remarked by Ramarkrishman,[2] an analytic continuation is possible to the case of "spacelike vectors": Formula (2) can be rewritten as

$$\tanh(A+B) = \frac{\coth A + \coth B}{1 + \coth A \cdot \coth B} \tag{5}$$

so that the two expressions (4) are exchanged. However, the third vector must remain timelike, because $w = \coth C$ is not a faithful mapping of the real line. Thus, as explained by Ramakrishnan, "superluminal Lorentz transformations" are excluded.

Formally speaking, all this is quite general, not committed to the relativistic composition of translations, the language of which we have used merely for brevity.

In the special relativity case, Lévy-Leblond[5,6] has termed A a "rapidity," so that, as shown by formulas (3), "composition of velocities is addition of rapidities."

Ramakrishnan has aimed his demonstration at special relativity, where it has concluded a long and very interesting quest, the history of which I will now very briefly outline.

3. FROM FRESNEL'S ETHER DRAG FORMULA TO THE RELATIVISTIC VELOCITY COMPOSITION FORMULA

Fresnel's ether-drag formula was tailored so as to "juggle away" the ether-wind effect to first order in $\beta = v/c$; the latter point he made explicitly.

In 1852 Fizeau explored experimentally the velocity dependence of the law by means of a laboratory setup.

In 1874, using Fermat's extremum principle, Potier[7] proved that Fresnel's formula has the general consequence of suppressing all first-order ether-wind effects. He adduced also the remark that Fresnel's formula truly is a velocity composition law.

In 1907, M. von Laue[8] showed that Fresnel's formula merely specifies, in differential form, the relativistic velocity composition formula as given by Poincaré and by Einstein in their seminal relativity papers.

In 1930 Hadamard,[9] using Lie's group theory, obtained the relativistic velocity composition formula by integrating Fresnel's formula.

Finally, in 1973, Ramakrishnan[3] put the whole matter in the pill form which has been recalled in Section 2.

Of course, a more detailed account of the story would bring in other names, such as Stokes, Veltmann, Abelé,[10] and Malvaux[11] (who "rediscovered" in 1952 Hadamard's reasoning) and Lévy-Leblond[5] and co-workers,[6] who gave high-brow presentations of the argument.

4. COULD FRESNEL HAVE ARRIVED AT THE RELATIVISTIC VELOCITY COMPOSITION FORMULA?

Fresnel's ether-drag formula reads

$$c' = \frac{c}{n} + v(1 - 1/n^2) \tag{6}$$

with c denoting the velocity of light in vacuo, c/n its velocity in a medium of index n and velocity v, and c' its resulting velocity in the laboratory. Setting

$$u = 1/n, \qquad u + \Delta u = c'/c$$

we write

$$v/c = \Delta u/(1 - u^2) \tag{7}$$

Then, as both v/c and Δu are small with respect to 1, to u, and to $1 - u$, we consider them as differentials, and rewrite (7) as

$$dA = du/(1 - u^2) \tag{8}$$

which is the differential of (1).

The meaning is that the physical step leading from c/n to c' is not the subtraction $c' - c/n$, but the composition (8).

So, there is no doubt that these clever people—Fresnel, Stokes, Potier—could have taken the step to special relativity. But, of course, some time was needed for a seasonal change and a ripening of the idea. The shocks of Michelson's null result and of the Fitzgerald–Lorentz specially concocted formula were quite instrumental in this. However, it is worth remarking that, in his seminal 1905 article, Einstein did not mention Michelson's experiment, but did mention Fresnel's ether-drag formula.

5. BRIEF CONCLUSION

The significance of Ramakrishnan's demonstration may well extend beyond the two cases of composition of translations (for which it was derived) and of rotations (where Carmeli has used it).

APPENDIX 1

We seek a generalization of the classical additive law (3) that is symmetric and departs minimally from it when A, B, and C depart from zero. Thus we write, in first order, c denoting a constant,

$$u + v + w \pm c^{-2} uvw = 0$$

By changing the scale we set $c = 1$.

Setting $v = -u - du$ we get $w = du/(1 \mp u^2)$, that is, w being small, $dA = du/(1 \mp u^2)$. The solution $u = \tan A$ is not satisfactory, but $u = \tanh A$ is.

APPENDIX 2

The reasons why angular velocities of rotating extended bodies are upper bounded should be explored. Of course the rim's linear velocity cannot exceed c, and also its radius is stretched by the centrifugal force. But Carmeli's data suggest that there is a more stringent limitation.

REFERENCES

1. M. Carmeli, *Found. Phys.* **15**, 889 (1985).
2. M. Carmeli, *Found. Phys.* **15**, 1263 (1985).
3. A. Ramakrishnan, *J. Math. Anal.* **42**, 377 (1973).
4. H. Yilmaz, in *Proceedings of the Berkeley Colloquium on the Philosophy of Science* (D. Reidel, Dordrecht, 1972).
5. J. M. Lévy-Leblond, *Am. J. Phys.* **44**, 271 (1976).
6. J. M. Lévy-Leblond and J. P. Provost, *Am. J. Phys.* **47**, 1045 (1976).
7. A. Potier, *J. Phys. (Paris)* **18**, 422 (1874).
8. M. von Laue, *Ann. Phys. (Leipzig)* **23**, 989 (1907).
9. J. Hadamard, *Cours d'Analyse professé à l'Ecole Polytechnique* (Gauthier-Villars, Paris, 1930), Vol. 2, pp. 335–339.
10. J. Abelé, *C. R. Acad. Sci. (Paris)* **235**, 1007 (1952).
11. P. Malvaux, *C. R. Acad. Sci. (Paris)* **235**, 1009 (1952).