Foundations of Physics, Vol. 22, No. 12, 1992

Electromagnetic Gauge as an Integration Condition: De Broglie's Argument Revisited and Expanded

O. Costa de Beauregard[1]

Received January 13, 1992; revised April 15, 1992

Einstein's mass-energy equivalence law, argues de Broglie, by fixing the zero of the potential energy of a system, ipso facto *selects a gauge in electromagnetism. We examine how this works in electrostatics and in magnetostatics and bring in, as a "trump card," the familiar, but highly peculiar, system consisting of a toroidal magnet m and a current coil c, where none of the mutual energy W resides in the vacuum. We propose the principle of a crucial test for measuring the fractions of W residing in m and in c; if the latter is nonzero, the (fieldless) vector potential has physicality. Also, using induction for transferring energy from the magnet to a superconducting current, we prove that W is equipartitioned between m and c.*

1. INTRODUCTION

Quite independently of his[(1)] massive photon theory (antedating Proca's and Kemmer's presentations of the spin-1 particle theory), Louis de Broglie argued[(2)] that the electromagnetic 4-potential has physicality, its Lorentz or Lienard–Wiechert gauge being the good one. Essentially, the argument boils down to this: Einstein's mass-energy equivalence law, by fixing the zero of the potential energy of an electromagnetic system, *ipso facto* selects the gauge.

In Sections 2 and 3 we carefully examine how this argument works in the electrostatics of point charges and in the magnetostatics of current loops, respectively. In these two cases a stressed structure is needed for holding the system.

Section 4 then brings in a trump card, a self-static system the mutual energy W of which has topological invariance with respect to arbitrary dis-

[1] Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75006 Paris, France.

0015-9018/92/1200-1485$06.50/0

placements or deformations of its two material elements: interlaced toroidal magnet m trapping its flux Φ and current coil c of fixed intensity I. The interaction energy $W = nI\Phi$ definitely resides *entirely inside m and/or c* (none of it in the vacuum). Therefore, by weighing separately m and c, the fractions of W belonging to each can be measured. As W_m depends upon the magnetic field of the current while W_c depends upon the fieldless vector potential of the magnet, if W_c is measured as nonzero, physicality of the vector potential will be proved experimentally. This we discuss in Section 5.

In Section 6, using induction for transferring energy from the magnet to a superconducting current, we definitely *prove* that *the mutual energy $I\Phi$ is equipartitioned between magnet and current.*

Section 7 entitled *electromagnetic gauge as an integration condition* synthesizes the whole matter, the keystone of which is Einstein's mass-energy equivalence law. Incidences of physicality of the 4-vector potential upon the electromagnetic spacetime stress tensor are discussed elsewhere.[3]

Four Appendices discuss related matters: Vaschy's[4, 5] theorem, hidden momentum in current loops,[6–8] self-energy of toroidal magnets, and the Aharonov–Bohm effect.

2. ELECTROSTATICS OF POINT CHARGES

Via Einstein's mass-energy equivalence law, the electrostatic mutual energy of point charges,

$$W = \sum\sum r^{-1} QQ' = (1/2)\sum V'Q + \text{const} \tag{1}$$

contributes to the rest mass of the system, which contribution goes to zero if the mutual distances go to infinity. Therefore, *if* the mutual energy W of the system is expressed as residing inside the point charges, *then* the constant in Eq. (1) must be zero, which means that the Coulomb gauge is uniquely selected.

If, in a further step, we conceive W as residing inside the point charges, *then* we have to inquire about the barycenter. A stressed structure is needed for ensuring that the system is static; if there are just two charges Q and Q', a uniformly tensed rod of length r will counterbalance the opposite Coulomb forces; per length unit it contains a uniform energy density $r^{-2}QQ'$: therefore, the barycenter of the mutual energy W is situated just at the middle between Q and Q', and we must write

$$W = \sum(1/2)V'Q \tag{2}$$

rather than $(1/2)\sum V'Q$: *the mutual energies are equipartitioned between the charges.*

Although it is outside the scheme of point charges, the question of "the radius of the electron" is amenable to similar considerations. The "Poincaré pressure" is the stress needed for holding together a charged spherical sphere of radius r; assuming that its rest mass entirely comes from the self-energy e^2/r yields the "classical electron radius."

3. MAGNETOSTATICS OF CURRENT LOOPS

Via Einstein's mass-energy equivalence law, the mutual energy

$$W = II' \oint\oint r^{-1} \, \mathbf{dl} \cdot \mathbf{dl}' = (1/2) \sum I \oint (\mathbf{A}' + \partial U') \cdot \mathbf{dl} \tag{3}$$

of an array of current loops contributes to the rest mass of the system.

A stressed structure is needed for holding it together. If there are two circular currents I and I' of common axis and same radius, a uniformly tensed tubular rod will counterbalance the opposite Laplace forces; this shows that the barycenter of the mutual energy is just at the middle between the circles.

This being said, we must discard the term $\partial U'$ inside formula (3) because, as an easy calculation shows, its presence could displace the barycenter of the mutual energy by an arbitrary amount, which is physically unacceptable.

We thus conclude that the mutual energy of a system of permanent current loops must be written as

$$W = \sum (1/2) I \oint \mathbf{A}' \cdot \mathbf{dl} \tag{4}$$

rather than as $(1/2) \sum I \oint \mathbf{A}' \cdot \mathbf{dl}$, with no gradient contribution. That is, *if* we *conceive* and *express* the mutual energy W of a system of Amperian currents as localized inside them, *then*, via Einstein's mass-energy equivalence law, we conclude first that *it is equipartitioned* between them, and second that *the Ampère gauge of the vector potential* such that

$$A = I \oint r^{-1} \, \mathbf{dl} \tag{5}$$

is uniquely selected.

4. INTERLACED TOROIDAL MAGNET AND CURRENT LOOP

This familiar system is a highly peculiar one, the idiosyncrasies of which will be displayed now.

First we make clear that a toroidal magnet m trapping its flux Φ need not be a torus *stricto sensu*; it can be a topological torus, here idealized as a closed wire each line element $\mathbf{dl}$ of which carries a magnetic moment $\mathbf{dM} = \Phi\, \mathbf{dl}$, with Φ constant. Such a magnet can be thought of as the limit of a closed chain of identical dipoles; its induction $\mathbf{B} = \partial \times \mathbf{A}$ is "distributed" along the loop, and (as there are no external poles) its field $\mathbf{H}$ is zero everywhere. Thus, it is surrounded by a curl-less vector potential A.

In the system we consider, such a toroidal magnet m is interlaced n times with a closed current loop c of intensity I. In the familiar version, m is a torus *stricto sensu* threaded n times by a current coil c. In the other extreme case the current loop c is a torus *stricto sensu* threaded n times by a magnetized coil m. In the general case, however, what we have is a "double helix," outside of which the two interlaced loops are closed.

The peculiarity of this system is topological invariance of its mutual (or interaction) energy

$$W = nI\Phi, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6}$$

I denotes the trapped electric current, Φ the trapped magnetic flux, and n the number of mutual twists, helicity sign included.

As a corollary to the topological invariance of W, a unique property of this system is that *it is self-static*: no stressed structure is needed for holding m and c in their relative positions, which are completely arbitrary. Of course both m and c are internally stressed by their self-energies; but the mutual energy W causes no mutual stresses. Then, via Einstein's mass-energy equivalence law, follows a unique possibility: *weigh separately m and c, thus ascribing to each definite fractions of the mutual energy W, W_m and W_c.*

There is no contribution to the mutual energy from the vacuum surrounding the system (because, since the magnet's field is identically zero, the corresponding term in Maxwell's stress tensor is missing). Thus, there exists the possibility of weighing *directly* the mutual or interaction energy by placing the whole system on the pan of a balance, and reversing the sign of either the flux Φ or the intensity I; this conserves both self-energies, but reverses the sign of the mutual energy $W \equiv W_m + W_c$.

The field equations are needed at this point. First, $\mathbf{H}$ denoting the current's magnetic field, "Ampère's theorem" applied to the magnet's loop threaded n times by the intensity I yields

$$\oint \mathbf{H} \cdot \mathbf{dl}_m = nI \tag{7}$$

which, combined with (6), expresses the mutual energy in the form

$$W = \Phi \oint \mathbf{H} \cdot \mathbf{dl}_m \tag{8}$$

that is, as residing inside the magnet (remember that $\Phi\, \mathbf{dl}_m$ is the magnetic moment per line element). Second, the "flux theorem" applied to the current loop threaded n times by the magnetic flux Φ yields, $\mathbf{A}$ denoting the (curl-less) vector potential created by the magnet,

$$\oint \mathbf{A} \cdot \mathbf{dl}_c = n\Phi \tag{9}$$

whence, in combination with (6), an expression of the mutual energy as residing inside the current loop:

$$W = I \oint \mathbf{A} \cdot \mathbf{dl}_c \tag{10}$$

The mutual symmetry between formulas (8) and (10) is conspicuous. Either of them yields the correct contribution $c^{-2}W$ to the total mass of the system, but inquiring about the barycenter will need more refinement. By inserting *either* in (7) the expression

$$\mathbf{H} = I \oint r^{-3}\mathbf{r} \times \mathbf{dl}_c \tag{11}$$

of the magnetic field generated by the current loop *or* in (9) the very similar expression

$$\mathbf{A} = \Phi \oint r^{-3}\mathbf{r} \times \mathbf{dl}_m \tag{12}$$

of the vector potential generated by the magnet, we get the formula

$$W = I\Phi \oint\oint r^{-3}\mathbf{r} \cdot [\mathbf{dl}_c \times \mathbf{dl}_m] \tag{13}$$

which is quite symmetric in m and c.

Formula (12) *with no additive gradient* follows straightaway from (8), (10), and (13) taken in this order; it is the one displaying the Ampère gauge of the vector potential.

As a curiosity, we derive from Eqs. (6) and (13) the following expression, valid in Euclidean space, for the integer number of mutual twists of m and c (the meaning of which is topological):

$$n = \oint\oint r^{-3}\mathbf{r} \cdot [\mathbf{dl}_c \times \mathbf{dl}_m] = 0, \pm 1, \pm 2, \ldots$$

Invariance of the system under exchange of m and c is conspicuous.

5. A CRUCIAL (BUT DIFFICULT) TEST OF PHYSICALITY OF THE POTENTIAL

Rather than literally weighing separately the interlaced magnet m and current loop c (which is possible), we prefer an alternative procedure using the equivalent energies. According to formula (13),

$$\mathbf{R} = n^{-1} \oint\oint \mathbf{r}\{\mathbf{r} \cdot [\mathbf{dl}_c \times \mathbf{dl}_m]\} \tag{14}$$

expresses the distance between the geometric barycenters of m and c, that is, the distance between the barycenters of the fractions W_m and W_c of the mutual energy W. If $W_m = W_c = W/2$, the barycenter of W is invariant under exchange of m and c, and invariant also under reversal of W. If not, we can measure the fraction $a = (W_m - W_c)/2W$ by placing our system inside a uniform gravity field of strength $\mathbf{g}$ and reversing the sign of W; due to work with or against gravity, the energy exchange will be

$$2W_1 = 2W(1 + ac^{-2}\mathbf{R} \cdot \mathbf{g}) \tag{15}$$

rather than just $2W$. Thus, we can (in principle) measure a, that is, the degree of physicality or nonphysicality of the vector potential. The two extreme cases are

$$\begin{aligned} W_c = 0, \quad W_m = 1: &\quad \text{complete nonphysicality of } \mathbf{A} \\ W_c = W_m = 1/2: &\quad \text{full physicality of } \mathbf{A} \end{aligned} \tag{16}$$

While the first issue would confirm the overwhelmingly prevailing opinion, the second one would be more in harmony with the mathematical and physical symmetries of our system, and more in harmony also with the conclusions drawn in the preceding sections. In particular, if the toroidal magnet were replaced by a solenoid, formula (4), implying equipartition of the mutual energy, would hold.

6. INDUCTION, SUPERCONDUCTIVITY, AND EQUIPARTITION OF THE MUTUAL ENERGY

Using induction for transferring energy inside our system, and superconductivity for avoiding dissipation, we prove that *the mutual energy W resides exactly half in the magnet and half in the current.*

Assuming that initially m is a ferromagnet in its unmagnetized metastable state of latent energy W_0, and that c is a currentless superconducting loop, m contributes $c^{-2}W_0$, and c zero, to the total rest mass.

If, by symmetry breaking, m goes into a magnetized state of trapped flux Φ and self-energy W_m, a supercurrent of intensity I and self-energy W_c is induced in c. As the magnet's transition generates an electric field, but no magnetic field, no energy is radiated away from the system, so that the energy difference $W_0 - W_m$ is converted into the sum of W_c and the mutual energy W, and energy conservation is expressed as

$$W_0 = W_m + W_c + W \tag{17}$$

Now, an induced supercurrent expels the magnetic flux: $\Phi + \Phi_c = 0$, Φ denoting the inducing flux and Φ_c the induced flux. As $W = I\Phi$ and $W_c = (1/2)I\Phi_c$, *the self-energy of a supercurrent equals minus half its mutual energy with the sources of the field*:

$$W + (1/2)W_c = 0 \tag{18}$$

In our case, from (17) and (18) there follows

$$W + (1/2)W_m = W_0 \tag{19}$$

which, together with (18), formalizes the announced statement.

7. CONCLUSION: ELECTROMAGNETIC GAUGE AS AN INTEGRATION CONDITION

de Broglie's argument, revisited and expanded as explained above, makes quite plausible that *the electromagnetic gauge is selected, via Einstein's mass-energy equivalence law, as an integration condition*; also, that *the mutual energies of a static system are equipartitioned between its material constituents*. The Coulomb gauge in electrostatics, and the Ampère gauge in magnetostatics, are thus selected.

If so, the linear energy densities displayed in formulas (4) and (10) have local physical meaning. Also, inside a current loop immersed in the fieldless vector potential created by a toroidal magnet resides physically half the mutual energy of the system.

Then a current loop with a finite section must physically contain an energy density $(1/2)\mathbf{A} \cdot \mathbf{j}$. This raises the question of the admissible spacetime electromagnetic stress tensors, a question discussed elsewhere.[3]

APPENDIX 1. VASCHY'S THEOREM

Vaschy's theorem is quoted without bibliographical reference by L. de Broglie[4] and by Jouguet[5] as stating this: "The integrated mutual energy

$\mathbf{H}_m \cdot \mathbf{H}_c$ between a Coulombian magnet (the field of which is a gradient) and an Amperian current (the field of which is a curl) is zero." While the *proof* is elementary and indisputable, the *wording* is quite questionable, because the mutual energy between a magnet (either Coulombian or Amperian) and a current loop is *certainly* nonzero.

Therefore, what Vaschy's theorem does say is that *the mutual energy of a Coulombian magnet and an Amperian current is not expressible in terms of Maxwell's stress tensor.* The conclusion to be drawn then is that the Coulombian picture of magnetism is hardly compatible with the concept of current loops.

APPENDIX 2. HIDDEN MOMENTUM IN CURRENT LOOPS

Here is a concise derivation of this phenomenon independently uncovered by Shockley–James,[6] Penfield–Haus,[7] and myself.[8]

If the intensity I of a current loop is slowly varied, the induced electric field confers to a point charge Q a momentum $\mathbf{dp} = Q\,\mathbf{dA}$; then the current loop must recoil by receiving the momentum $-\mathbf{dp}$:

$$\mathbf{dp} = Q\,dI \oint r^{-1}\,\mathbf{dl} = Q\,\mathbf{dA} = -c^{-2}\,dI \oint V\,\mathbf{dl} \tag{15}$$

$\mathbf{A}$ denotes the vector potential created by the current and V the scalar potential created by the charge, both gauge invariant. However, *local* "equality of action and reaction" holds if and only if the Coulomb and the Ampère gauges are used in combination, as above.

APPENDIX 3. SELF-ENERGY OF A TOROIDAL MAGNET

Maxwell's energy density $\mathbf{B} \cdot \mathbf{H}$ is identically zero inside and outside a toroidal magnet (so is $\mathbf{E} \cdot \mathbf{D}$ in and out a toroidal dielectric). Should we then conclude that a toroidal magnet has no self-energy? Besides being unbelievable, this would contradict Ampère's equivalence law between magnets and solenoids: the self-energy of a solenoid is $(1/2)nI\Phi$, and the trapped density of it is $(1/2)\mathbf{H}^2$.

Here again we find an incompatibility between Coulomb's and Ampère's pictures of magnetism. According to Coulomb the self-energy m^2/r of an infinitely thin and long magnet is zero, and the same is true for a circular magnetized wire; according to Ampère, however, the self-energy of such a wire is infinite!

Ampère's principle entails that the self-energy density inside a toroidal magnet should be $(1/2)\mathbf{B}^2 = (1/2)\mathbf{M}^2$, and that the magnetic polarization current circles around the torus. Can we justify this better than by an *ad hoc* argument?

In an "invited paper" twin to this one,[3] it is argued that the total electromagnetic energy-momentum density, *sources of the field included*, is a sum of tensors not separately gauge invariant, the trace of which is $(1/4)B^{kl}B_{kl} + (1/2)A^k j_k$, with $B^{kl} = H^{kl} + M^{kl}$.

Thus, the riddle of the self-energy of a toroidal magnet, seemingly unsolvable within the accepted paradigm where the vector potential has no physicality, provides one more argument favoring de Broglie's views.

APPENDIX 4. LORENTZ CONDITION AND THE DIRAC ELECTRON THEORY

The Lorentz condition, a very strong restriction upon gauge invariance, requires that the arbitrary superpotential U obey the sourceless d'Alembert equation, and thus propagate as a vacuum field magnitude. Its Fourier expansion consists of the so-called (lightlike) "longitudinal plane waves."

The Lorentz condition is inherent in the Dirac electron theory, as it is necessary for deriving the second-order equation. Significant consequences (not yet emphasized as it seems) then follow, *which the Schrödinger equation did not imply.*

For a Dirac electron immersed in an electrostatic field, such as the hydrogen atom field, the additive energy constant is the gradient of an x-independent $C\exp(ivt)$ function, which must be zero to satisfy the Lorentz condition. Thus, *as a consequence of relativistic invariance, the Coulomb gauge is uniquely selected.*

Of course, an x-independent additive constant C is acceptable if the gauge field is restricted to the inside of a surface enclosing the atom, for example, a sphere of radius R; C must then be R dependent, and go to zero if R goes to infinity. A natural substitute to C then is $Q = CR$, the surface charge creating the potential C; adding to C one more constant would only initiate an infinite regress.

For a Dirac electron immersed in a magnetostatic field, such as the one undergoing an Aharanov–Bohm effect, the time independent $U(\mathbf{x})$ is not arbitrary, but most obey the sourceless Laplace equation. In terms of Fourier components, this is expressed as $\mathbf{k} = 0$. Thus, the electron's potential momentum $e\mathbf{A}$ is asymptotically zero and, *as a consequence of relativistic invariance, the Ampère gauge is uniquely selected.*

The recipe for gauge invariance of the Dirac equation is independent of the field-generating 4-potential, so it holds for the free electron; enforcing it there would superpose on the solution under study an (irrelevant) extreme-relativistic electronic wave, which is tacitly discarded.

This is reminiscent of what occurs in the electromagnetic field *per se*, where the vacuum gauge waves are the so-called longitudinal plane waves. These, carrying neither energy nor angular momentum, are tacitly left aside.

These remarks illustrate the theme *electromagnetic gauge as an integration condition*, and aim at conveying the impression that there may be some physics in the choice of a gauge. In particular, the Lorentz condition has so many good consequences that one easily feels that it has some physical meaning.

Of course, potentials never enter the expression of *forces*, but they do enter that of their space and time integrals, *energy* and *momentum*; thus, it is only natural that *integration conditions* confer physical values to the potentials.

NOTE ADDED IN PROOF

Coleman and Van Vleck's paper[9] on "Hidden Momentum in Magnets" contains implicitly[10] a striking argument in favor of the views presented here.

REFERENCES

1. L. de Broglie, *Une Nouvelle Conception de la Lumière* (Hermann, Paris, 1934); *Nouvelles Recherches sur la Lumière* (Hermann, Paris, 1936).
2. L. de Broglie, *Compt. Rend. Acad. Sci.* **225**, 163 (1947).
3. O. Costa de Beauregard, *Ann. Fond. Louis de Broglie*, in press.
4. L. de Broglie, *Portugaliae Physica* **3**, 1–19 (1949); see p. 11.
5. M. Jouguet, *Cours de Physiqe, Vol. 2. Electricité* (Eyrolles, Paris, 1962), p. 224.
6. W. Shockley and R. P. James, *Phys. Rev. Lett.* **18**, 876 (1967).
7. H. Haus and P. Penfield, *Phys. Rev. Lett. A* **26**, 412 (1968).
8. O. Costa de Beauregard, *Phys. Rev. Lett. A* **24**, 177 (1967).
9. S. Coleman and J. H. Van Vleck, *Phys. Rev.* **171**, 1370 (1968).
10. O. Costa de Beauregard, *Phys. Lett.* **28A**, 365 (1968).