Statics of filaments and magnetostatics of currents: Ampère tension and the vector potential

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The challenge of understanding Saumon's experimental findings has motivated this derivation of magnetostatics from the expression \( T = iA \) of the stress tension along a stiff current loop. From the integral equivalence of the Ampère and Laplace forces we derive the existence of a repulsive tension \( T = iA \) inside a straight current.

1. Magnetostatics of current loops derived from the statics of filaments

As is well known, the force density \( f \) along a (possibly stiff) filament of line element \( dl \) is related to the tension \( T \) via
\[
f = dT/dl;
\]
also, per line element the torque
\[
dC = T \times dl
\]
is applied; finally (think for example of a coil-spring holding a weight in a gravity field) a potential energy
\[
dW = T \cdot dl
\]
is stored per line element.

At the extremities \( P \) and \( Q \) of a filament in static equilibrium, tensions \( T_P \) and \( T_Q \) counterbalancing the force and torque densities distributed along it, build up a representation of the corresponding wrench, or combined force and torque [1].

Consider then a rigid current loop of intensity \( i \), at equilibrium with its magnetostatic field; if ideally severed at two points \( P \) and \( Q \), the opposite wrenches applied to the complementary segments \( PQ \) and \(QP \) are represented by paired opposite tensions \( \pm T_P \) and \( \pm T_Q \). We will show that these are tangent to the filament, and exactly expressed in the form
\[
T = iA
\]
where \( A \) denotes the vector potential in the Ampère gauge.

\[
dA = -r^{-1} i dl.
\]

Postponing the proof, we now use these two premises for a concise derivation of the well known formulas of the magnetostatics of current loops.

Substituting (4) in (3) we get the gauge dependent differential expression of the magnetic energy
\[
dW = iA \cdot dl,
\]
the closed integral of which is the familiar \( i \Phi \); substituting (5) we get the action-at-a-distance style expression of the mutual energy of two current elements,
\[
d_2 W = r^{-1} i r' dl \cdot dl',
\]
a factor \( i \) intruding in the double summations.

From (1), (4), (5) and the identity \( \partial r^{-1} = -r^{-3} \partial r \) we get
\[
d_2 T = \pm r^{-3} i r' \partial (dl \cdot dl') r
\]
as expressing the directly opposite shortened Ampère forces felt by two current elements, well known to be integrally equivalent to the Laplace forces.

Finally, inserting (4) into (2) we get the gauge dependent expression of the torque applied to a current element,
\[
dC = iA \times dl,
\]
and, substituting (5), the action-at-a-distance expression of the opposite torques felt by two current elements,

\[ d_2 C = \pm r^{-1} i_1 i_2 d\vec{l}_1 \times d\vec{l}_2. \]  

(10)

Linear and angular action-reaction opposition between interacting current loops at mutual rest is thus expressed in instantaneous action-at-a-distance style.

Now we apply these formulas to two complementary arcs PQ and QP of the same circuit.

In the zero cross section limit, the integral of formula (5) diverges; the familiar cut-off procedure uses a phenomenological self-induction coefficient, a pure number \( k \) such that \( dW/dl = ki \); the \( r^{-1} \) dependence of \( A \) to its source entails that the Ampère gauged \( A \) is tangent to the circuit, with the value \( A = ki \). Thus \( ki \) is the linear energy density \( w = dW/dl \) and the longitudinal self-stress tension \( T = ki \), this definitely proves the existence of a longitudinal repulsive stress tension \( T = ki \) along any current carrying wire. It is of course quite usual that a trapped energy density exerts an equivalent pressure upon its container.

So, the paired opposite tensions \( \pm T_1 \) and \( \pm T_0 \) applied at severance points of a current loop are tangent to it; if expressed in the form \( iA \), no arbitrariness is allowed, so \( A \) must be taken in the Ampère gauge. This is exactly similar to what happens in, say, the electron self-energy problem, where the electric potential's Coulomb gauge must be used for expressing the self-energy \( e^2/2r \).

This completes the proof alluded to after formulas (4) and (5).


Consider the case where the segment PQ is a "hairpin" connecting two mercury containing cups inserted in a current loop. In Ampère's experiment, where the cups are at the ends of parallel "rails" [3], the "hairpin" is repelled by the Laplace-Lorentz force. The point is that the paired rails do feel the opposite reaction, that is, the Ampère compression. This is confirmed in an experiment of Graneau [3].

A quite significant experiment would use a Z-shaped rather than a U-shaped circuit, the hairpin bridging cups at the ends of antiparallel rails, and a torsion balance measuring the torque. The prediction is that this torque must accentuate rather than flatten the Z.

Saumont's device can be seen as a refinement of Ampère's one. Very consistent measurements using it, including counter tests, do display the repulsive Ampère tension in the form \( T = ki \), and show that it is longitudinal inside the mercury. The number \( k \) comes out as \( k \approx 2.45 \).

3. Concluding remarks

(1) Read in terms of the statics of filaments, the equations of the magnetostatics of current loops entail that the physical stress tension \( T \) along a (possibly stiff) conducting wire exactly is expressed as \( iA \), \( i \) denoting the intensity and \( A \) the vector potential in the Ampère gauge. This is exactly similar to what occurs in the electron self-energy problem: as expressed in terms of the electric potential \( V \), the correct value of the self-energy \( e^2/2r \) obtains iff the Coulomb gauge is selected.

Other examples [4] are easily adduced, all under the heading: electromagnetic gauge selected as an integration condition.

(2) The well known integral equivalence between the Ampère force proper, or a shortened expression of it, and the standard Laplace-Lorentz force, is thus confirmed; but it turns out that the concept of an Ampère tension \( T = iA \) is much preferable.

The said equations definitely entail the existence of a longitudinal repulsive stress tension \( T = ki \) along any current carrying wire: this being the integral rendering of the differential Laplace-Lorentz force.

(3) Saumont's [2] experiments, viewed as improving Ampère's so-called "hairpin experiment", do display the repulsive Ampère tension; also, they exemplify "selection of the electromagnetic gauge as an integration condition" for expressing action-reaction opposition.

References