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Statics of filaments and magnetostatics of currents: Ampère tension and the vector potential

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The challenge of understanding Saumont's experimental findings has motivated this derivation of magnetostatics from the expression $T=iA$ of the stress tension along a stiff current loop. From the integral equivalence of the Ampère and Laplace forces we derive the existence of a repulsive tension $T=ki^2$ inside a straight current.

1. Magnetostatics of current loops derived from the statics of filaments

As is well known, the force density f along a (possibly stiff) filament of line element $\mathrm{d}l$ is related to the tension T via

$$f=\mathrm{d}T/\mathrm{d}l\,; \tag{1}$$

also, per line element the torque

$$\mathrm{d}C=T\times\mathrm{d}l \tag{2}$$

is applied; finally (think for example of a coil-spring holding a weight in a gravity field) a potential energy

$$\mathrm{d}W=T\cdot\mathrm{d}l \tag{3}$$

is stored per line element.

At the extremities P and Q of a filament in static equilibrium, tensions T_P and T_Q, counterbalancing the force and torque densities distributed along it, build up a representation of the corresponding *wrench*, or combined force and torque [1].

Consider then a rigid current loop of intensity i, at equilibrium with its magnetostatic field; if ideally severed at two points P and Q, the opposite wrenches applied to the complementary segments PQ and QP are represented by paired opposite tensions $\pm T_P$ and $\pm T_Q$. We will show that these are tangent to the filament, and *exactly* expressed in the form

$$T=iA\,, \tag{4}$$

where A denotes the vector potential *in the Ampère gauge*,

$$\mathrm{d}A=r^{-1}i\,\mathrm{d}l\,. \tag{5}$$

Postponing the proof, we now use these two premisses for a concise derivation of the well known formulas of the magnetostatics of current loops.

Substituting (4) in (3) we get the gauge dependent *differential* expression of the magnetic energy

$$\mathrm{d}W=iA\cdot\mathrm{d}l\,, \tag{6}$$

the closed integral of which is the familiar $I\Phi$; substituting (5) we get the action-at-a-distance style expression of the mutual energy of two current elements,

$$\mathrm{d}_2W=r^{-1}ii'\,\mathrm{d}l\cdot\mathrm{d}l'\,, \tag{7}$$

a factor $\frac{1}{2}$ intruding in the double summations.

From (1), (4), (5) and the identity $\partial r^{-1}=-r^{-3}r$ we get

$$\mathrm{d}_2T=\pm r^{-3}ii'(\mathrm{d}l\cdot\mathrm{d}l')r \tag{8}$$

as expressing the *directly opposite shortened Ampère forces* felt by two current elements, well known to be *integrally equivalent to the Laplace forces*.

Finally, inserting (4) into (2) we get the gauge dependent expression of the torque applied to a current element,

$$\mathrm{d}C=iA\times\mathrm{d}l\,, \tag{9}$$

and, substituting (5), the action-at-a-distance expression of the *opposite torques* felt by two current elements,

$$d_2C = \pm r^{-1} ii' \, dl \times dl' \, . \tag{10}$$

Linear and angular action–reaction opposition between interacting current loops at mutual rest is thus expressed in instanteneous action-at-a-distance style.

Now we apply these formulas to two complementary arcs PQ and QP of the same circuit.

In the zero cross section limit, the integral of formula (5) diverges; the familiar cut-off procedure uses a phenomenological self-induction coefficient, a pure number k such that $dW/dl = ki^2$; the r^{-1} dependence of $\boldsymbol{A}$ to its source entails that the Ampère gauged $\boldsymbol{A}$ is tangent to the circuit, with the value $A = ki$. Thus ki^2 is the common value of the linear energy density $w = dW/dl$ and of the longitudinal self stress-tension $T = iA$; *this definitely proves the existence of a longitudinal repulsive stress tension* $T = ki^2$ *along any current carrying wire*. It is of course quite usual that a trapped energy density exerts an equivalent pressure upon its container.

So, *the paired opposite tensions* $\pm \boldsymbol{T}_P$ and $\pm \boldsymbol{T}_Q$ *applied at severance points of a current loop are tangent to it; if* expressed in the form $i\boldsymbol{A}$, *no arbitrariness is allowed, so* $\boldsymbol{A}$ *must be taken in the Ampère gauge*. This is exactly similar to what happens in, say, the electron self-energy problem, where the electric potential's Coulomb gauge *must* be used for expressing the self-energy $e^2/2r$.

This completes the proof alluded to after formulas (4) and (5).

2. Hairpin style devices and Saumont's [2] experiments

Consider the case where the segment PQ is a "hairpin" connecting two mercury containing cups inserted in a current loop. In Ampère's experiment, where the cups are at the ends of parallel "rails" [3], the "hairpin" is repelled by the Laplace–Lorentz force. *The point is that the paired rails do feel the opposite reaction*, that is, the Ampère compression. This is confirmed in an experiment of Graneau [3].

A quite significant experiment would use a Z-shaped rather than a U-shaped circuit, the hairpin bridging cups at the ends of antiparallel rails, and a torsion balance measuring the torque. The prediction is that *this torque must accentuate rather than flatten the Z.*

Saumont's device can be seen as a refinement of Ampère's one. Very consistent measurements using it, including counter tests, do display the repulsive Ampère tension in the form $T = ki^2$, and show that it is longitudinal inside the mercury. The number k comes out as $k \approx 2.45$.

3. Concluding remarks

(1) Read in terms of the statics of filaments, the equations of the magnetostatics of current loops entail that the *physical* stress tension $\boldsymbol{T}$ along a (possibly stiff) conducting wire *exactly* is expressed as $i\boldsymbol{A}$, i denoting the intensity and $\boldsymbol{A}$ the vector potential in the Ampère gauge. *This is exactly similar to what occurs in the electron self-energy problem*: as expressed in terms of the electric potential V, the correct value of the self-energy $e^2/2r$ obtains *iff* the Coulomb gauge is selected.

Other examples [4] are easily adduced, all under the heading: *electromagnetic gauge selected as an integration condition.*

(2) The well known integral equivalence between the Ampère force proper, or a shortened expression of it, and the standard Laplace–Lorentz force, is thus confirmed; but it turns out that *the concept of an Ampère tension* $\boldsymbol{T} = i\boldsymbol{A}$ *is much preferable.*

The said equations definitely entail *the existence of a longitudinal repulsive stress tension* $T = ki^2$ *along any current carrying wire, this being the integral rendering of the differential Laplace–Lorentz force.*

(3) Saumont's [2] experiments, viewed as improving Ampère's so-called "hairpin experiment", do display the repulsive Ampère tension; also, they exemplify "selection of the electromagnetic gauge as an integration condition" – for expressing action–reaction opposition.

References

[1] B. Hoffmann, About vectors (Prentice-Hall, Englewood Cliffs, NJ, 1966) pp. 83, 84.
[2] R. Saumont, Phys. Lett. A 28 (1968) 365, and private communications.
[3] P. Graneau, J. Phys. D 20 (1987) 391.
[4] O. Costa de Beauregard, Found. Phys. 22 (1992) 1485.