ELECTROMAGNETIC GAUGE AS INTEGRATION CONDITION: EINSTEIN'S MASS-ENERGY EQUIVALENCE LAW AND ACTION-REACTION OPPOSITION

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To the memory of Louis de Broglie,
dedicated supporter of the potential.

Abstract

Much more physics is implied in the choice of a gauge than is generally admitted. Einstein's mass-energy equivalence law and action-reaction opposition (Newton's third law) in quasi-static systems are two gravitationally-induced conditions placing imperative restrictions on equations displaying the electric or the magnetic potential. Familiar (but overlooked) examples of this are recalled. Significant new ones are Graneau's and Saumont's experiments evidencing the Ampère stress tension $\mathbf{T} = I\mathbf{A}$ along a current loop, and through it physicality of the vector potential. Also, angular recoil of a variable dipole magnet immersed in a uniform electric potential is predicted.

No denial of the special relativity theory (SRT) is implied in these arguments, and indeed Section 11 extends this discussion to the relativistic far-action Wheeler-Feynman electrodynamics. Section 13 recalls de Broglie's definition of an electromagnetic spin density implying the 4-potential.

Section 16 presents a general argument pertaining to the Lorentz condition, a strong Lorentz-invariant restriction on the gauge *definitely required in any form of relativistic electrodynamics*. As the free-gauge-transforming Lorentz solutions are sourceless propagating field magnitudes, ghosts so to speak, source-adhering solutions obeying the Lorentz condition (such as Liénard-Wiechert's) are left as *the* physical solutions of Maxwell's equations. Section 17 extends the gauge discussion to the Dirac electron theory.

As the *wrench* concept, familiar in Euclidean-Galilean statics, kinematic and dynamics of solids, is no longer so familiar, and I have used it, Section 19 recalls it.

1. Introduction

Physicality of the electromagnetic potential is held as a heresy or truism, depending on zero or non-zero photon rest mass; so abrupt a discontinuity is questionable, and does not seem very physical. As in many problems, the solution is best presented in an adapted gauge; it is not certain that the choice of a gauge is just an affair of predilection, and that no physical justification is needed. Examples can be adduced to the contrary, a few significant ones being presented in this paper. Here are three simple ones offered as appetizers.

In the classical electron radius problem, expressing the self-energy $e^2/2r$ in terms of the electric potential as $-eV/2$, *requires* the use of the Coulomb gauge $V = -e/r$; no other choice is left.

This gauge is selected as an integration condition, via Einstein's mass-energy equivalence law. Stating that, in energy units, the electron's rest mass is some 511,000 electron-volts (strictly speaking, *positron-volts*) makes sense in the Coulomb gauge only. This shows that the choice of gauge and of integration condition are mutually implicative, both being dictated by the underlying physical model.

A related remark holds for the mutual energy: the weight of a vessel containing gaseous or liquid hydrogen includes the "atomic mass defect"; the additive constant in Coulomb's binding energy is thus weighed as zero.

As a second example, consider a permanent current loop of intensity I at equilibrium with its magnetostatic field. If the expression chosen for its self-energy is the (gauge-invariant) one,

$$W = \frac{1}{2}I\Phi \equiv \frac{1}{2}I\oint \mathbf{A}\cdot d\mathbf{l}\,, \tag{1}$$

there remains the fact that barycenter's position depends on the linear distribution of the vector potentials; therefore, at each point of the loop, $\mathbf{A}$ *must* have a definite value. Einstein's mass-energy equivalence dictates that a definite answer be found.

From the far-action expressed of the mutual energy of a pair of current elements,

$$d^2W = I^2 r^{-1} d\mathbf{l}\cdot d\mathbf{l}' \tag{2}$$

(where the factor 1/2 intrudes in a double summation), the needed expression is *uniquely* fixed as the Ampère gauge

$$\mathbf{A} = I\oint r^{-1} d\mathbf{l}\,. \tag{3}$$

Introducing our third example, we note that *in any thought experiment using weighing with a dynamometer, Einstein's gravity-inertia equivalence entails a mass (not rest mass) shift* — via "gravity redshift" on the matter wave of any massive particle. In a static Newtonian potential *proper* $U = -c^{-2}GM/R$, and m is thus changed into $m(1-U)$.

Consider then the timelike Aharonov-Bohm effect, the electric analog of the gravity lens effect. The phase shift $h^{-1}eVT$ displayed by re-superposing two separated electron beams having spent a time T each in a uniform potential of mutual difference V, displays the difference $-c^{-2}eV$ between effective masses.

Suppose that V is the potential difference between inside and outside a spherical capacitor held in a uniform gravity field $\mathbf{g}$. Measured outside the capacitor, an electron's weight certainly is the mechanical $m\mathbf{g}$; and certainly then, *V denoting the internal Coulomb potential proper*, the weight measured inside the capacitor is $(m - c^{-2}eV)\mathbf{g}$.

Using Wheeler-Feynman's relativistic electrodynamics, we will argue in Section 11 that *a point charge Q immersed in a given 4-potential A^i expressed in the source-adhering gauge is endowed with the extra-inertial 4-momentum QA^i*. This, together with Einstein's gravity-inertia equivalence, validates our repeated use of weighing thought experiments.

The Lorentz condition $\partial_i A^i = 0$, another central issue in this essay, *is inherent in any form of explicitly relativistic electrodynamics*, being a consequence of the inhomogeneous d'Alembert equation $\partial_i^i A^k = j^k$ and of the continuity equation $\partial_k j^k = 0$. From these, and the definitions of the 6-field and 6-polarization tensors, *the whole system of Minkowski-Maxwell equations necessarily follows.*

Concluding a review of various examples, a "high brow" argument penetrating to the Lorentz condition is articulated in Section 16, restoring continuity when going to the photon zero mass limit. *Being a differential law, gauge invariance can yield no objection against selecting the gauge as an integration condition.*

The writing of this paper was well advanced when T.W. Barrett called my attention to two works significantly advocating physicality of the electromagnetic potential: Konopinski's,[1] emphasizing the relevance of the vector potential in momentum exchanges, and his own,[2] based on SU(2) aspects of electromagnetism.

2. Electromagnetic Hidden Linear Momentum

An electrostatic system c and a magnetostatic system m both at rest have zero Maxwellian mutual energy: $\mathbf{E}_c\cdot\mathbf{E}_m + \mathbf{H}_c\cdot\mathbf{H}_m \equiv 0$. Also, neither force nor torque is exerted between them.

Nevertheless "hidden" opposite linear and angular momenta do reside in each — the fossils, so to speak, of the forces that were alive during the build-up of this self-static system. During that time a stressed structure was needed as a removable scaffolding.

Such systems are in a specific kind of indifferent equilibrium: moving one of the two pieces generates a transient force and/or torque that moves deterministically the other one. If the change is slow so that radiation is negligible the total work is zero, and so is the net energy balance. If the change is monitored from inside, the system's barycenter is conserved.

Such are the thought-provoking peculiarities implied in three independent[3-5] 1967–68 papers summarized in the next section, which have motivated subsequent comments.[6,7]

The point emphasized here is that, *as the opposite mutual momenta have expressions quadratic in a field and a potential, the electromagnetic gauge is fixed as an integration condition, via action-reaction opposition.*

As a simple example consider the system made up of a point charge c of value Q and a current loop m of intensity I and line element $\mathbf{dl}$, both at rest. The charge's well-known "hidden" momentum is

$$\mathbf{P}_c = Q\mathbf{A}, \qquad \mathbf{A} = I\oint r^{-1}\mathbf{dl}_m \,. \tag{4}$$

Let us show, via action-reaction opposition, that the current's hidden momentum is (in Gaussian units)

$$\mathbf{P}_m = I\oint V\mathbf{dl}, \qquad V = -r^{-1}Q \,; \tag{5}$$

$\mathbf{A}$ and V denote of course the vector and the scalar potential respectively created by the current and the charge. From formulas (4) and (5) we get the far-action expression

$$\mathbf{P}_c = -\mathbf{P}_m = QI\oint r^{-1}\mathbf{dl}, \qquad \text{QED.} \tag{6}$$

While the expression of the current's hidden momentum is gauge-invariant, the charge's is not; use of the Ampère gauge (3) is then required for expressing action-reaction opposition. Incidentally, a smaller remark was made long ago by J. J. Thomson.[8]

If, in the presence of a point charge c, a current's m intensity build up from zero to I, the loop recoils, and the charge is projected; this has been termed the *magnetodynamic effect*.[9]

These two opposite linear momenta being *not directly* so, there also exists a *hidden angular momentum* problem.

3. Electromagnetic Hidden Angular Momentum

As a preamble we ask if the elementary "hidden momentum" $VI\mathbf{dl}$ in the integral (5) has local meaning or not.

Consider for simplicity the system made up of a circular current of radius r with, at its center, a point charge generating the Coulomb potential proper $V = Q_0/r$. If the current's intensity $I = qv$ is produced by a rotating ring of linear charge density q, the moving stress-energy density $w = Vq$ has a tangential momentum $c^{-2}VI$ (esu); this remains true for a current running inside a conductor to which the stress tension $T = w$ is transferred, because the moving deformation imprinted upon the ring by each charge-carrier Q transports the linear momentum $c^{-2}VQ\mathbf{v}$.

In either case, $M = \pi r^2 I$ denoting the circuit's magnetic moment, *the "hidden angular momentum" $C = 2vM$ must have been conferred on the conducting loop* (2 is the value taken in the present case by a form factor we will meet with again).

In Sommerfeld's hydrogen atom theory, V denoting the Coulomb central potential proper, $c^{-2}eV\mathbf{v} = (1/2)m_0(v/c)^2\mathbf{v}$ are equivalent expressions of the orbiting electron's extra momentum; in a transition, its variation is included in the angular momentum exchanged with the photon field.

Correspondence between Sommerfeld's orbiting electron and an Ampèrian loop is thus verified. Potential momentum as a corollary to potential energy follows from mass-energy equivalence and relativistic covariance. Velocity dependence of the electron's "Maupertuisian mass" confers an anomalous gyromagnetic ratio on both Sommerfeld's atom, and an Ampèrian loop with a point charge at its center.

We thus conclude that *the "hidden linear momentum" $VI\mathbf{dl}$ has local meaning, and can in principle be tested in gyromagnetic experiments involving recoil of the conductor.*

This digression made, we return to the general problem.

Using in (5) the well-known[10] general integral transform identity, with $E_u = \partial_u V$ $(u = 1, 2, 3)$ denoting the charge's electric field, we express the current's linear momentum in the Poynting style

$$\mathbf{P}_m = I\iint \mathbf{E}\times\mathbf{ds} \,. \tag{7}$$

Using then Ampère magnetic-shell-current-loop equivalence, we express the linear momentum "hidden" in a dipole of magnetic moment $\mathbf{M}$ as

$$\mathbf{P}_m = \mathbf{E}\times\mathbf{M} \,. \tag{8}$$

Inserting in (4) the expression of the dipole's vector potential

$$\mathbf{A} = r^{-3}\mathbf{M}\times\mathbf{r} \tag{9}$$

and in (8) that of the charge's electric field

$$\mathbf{E} = -Qr^{-3}\mathbf{r} \,, \tag{10}$$

we recover action-reaction opposition in an action-at-a-distance style, QED.

So, the (not directly) opposite linear momenta, $\mathbf{P}_c$ attached to the charge and $\mathbf{P}_m$ attached to the dipole, build up a (CP-invariant) *hidden orbital angular momentum*

$$\mathbf{C}_0 = r^{-3}Q[\mathbf{M}\times\mathbf{r}]\times\mathbf{r} \equiv r^{-3}Q(\mathbf{M}\cdot\mathbf{r})\mathbf{r} - V\mathbf{M} \,, \tag{11}$$

the latter expression displaying the electric charge's potential V *in the Coulomb gauge*.

Suppose that the dipole, initially in a metastable unmagnetized state, spontaneously acquires later a magnetic moment **M**. The total (mechanical plus electromagnetic) linear and angular momentum must retain its initial value, zero. This we express as (6), and as

$$\mathbf{C}_0 + \mathbf{C}_m + \mathbf{C}_e = 0 \,, \tag{12}$$

where

$$\mathbf{C}_m = -V\mathbf{M}, \qquad \mathbf{C}_e = -r^{-3}Q(\mathbf{M}\cdot\mathbf{r})\mathbf{r} \,. \tag{13}$$

Thus we conclude that: (1) the point charge and dipole are projected in slingshot fashion with (not directly) opposite linear momenta; (2) the dipole acquires, besides its Einstein-de Haas mechanical spin, a "hidden spin" $\mathbf{C}_m$; and (3) it feels a precession momentum $\mathbf{C}_e$.

More of this in Section 12.

To conclude this section and the preceding one: expressed in coherent emu or esu units, the time-dependent mutual forces and torques discussed above are of order c^{-2}; so there is little surprise that their existence was only uncovered in 1967–68.

No denial whatsoever of the special relativity theory (SRT) is implied in the use we have made of instantaneous far-action-reaction in quasi-static systems; indeed Section 11, summarizing the Wheeler-Feynman electrodynamics, yields an essentially similar conclusion: *necessary selection of the source-adhering Liénard-Wiechert 4-potential for obtaining action-reaction opposition.*

4. Summary of Three 1967–68 Papers; Penfield-Haus Extra-Linear and Sommerfeld Extra-Angular Momentum

The argument in my paper[3] entitled "A new law in electrodynamics" was this: As relativistic invariance requires that a magnetic pole of charge R, feeling at rest a Coulomb force $R\mathbf{H}$, should feel in motion a Lorentz force $R(\mathbf{E}\times\mathbf{v}/c+\mathbf{H})$, a variable magnetic dipole of moment $\mathbf{M} = R\mathbf{a}$ in a constant electric field **E** must feel a force $\mathbf{E}\times d\mathbf{M}/dt$.

The source of **E** (for example a point charge) feels the transient electric field induced by the variation $d\mathbf{M}/dt$; one then verifies[3] that there is linear action-reaction opposition between the dipole and the point charge and, using Ampère's magnetic shell-current loop equivalence, one derives formula (5).

In Shockley's and James's paper[4] published soon after mine, "Try simplest cases discovery of hidden momentum forces on magnetic currents," the thinking is very heuristic, and not easily summarized. The conclusion they reach, and contemplate

testing, is existence of a force $\mathbf{E}\times d\mathbf{M}/dt$ acting upon a variable magnetic dipole immersed in a constant electric field.

Haus and Penfield,[5] in a paper entitled "Force on a current loop," review the whole matter, including their own previous thinking. From it I select an illuminating "Ampèrian derivation"[11] of formula (5) above, via the relativistic velocity dependence of the mass of conduction electrons.

At given intensity $I \equiv qv$, the linear charge density q and the current's velocity v depend at each point on the voltage V, because the ratio $-m/e$ of the electron's relativistic mass to its charge is velocity-dependent: $m = -c^{-2}eV+$ const. That each line element then carries a "hidden momentum" $IV\mathbf{dl}$ follows straightaway.

Penfield-Haus's argument is akin to Sommerfeld's in the hydrogen atom theory. But while Sommerfeld derives a relativistic extra-angular momentum displaying (as we have seen) the potential's Coulomb gauge, PH derive a relativistic extra-linear momentum implying only voltage *differences*.

Corollary:

Via Einstein's inertia-gravity equivalence at any intensity I, *including* $I = 0$, any conduction electron in a circuit at rest in a uniform gravity field has an extra weight $(mc^{-2}eV\mathbf{g})$; more of this in Sections 10 and 12.

5. Interlaced Toroidal Magnet and Current Loop

This very familiar system (Fig. 1) made up of two magnetostatic elements is even more thought-provoking[12] than the one previously discussed. This is a CP-invariant system, of SU(2) symmetry.

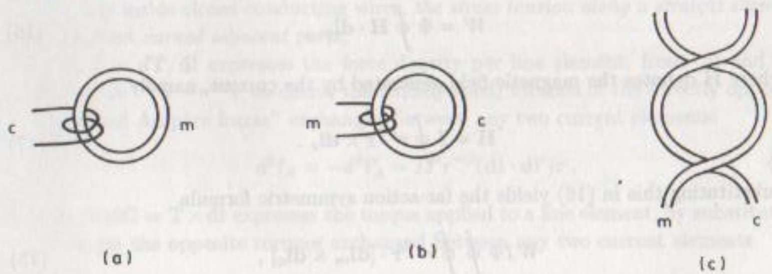

Fig. 1. Interlaced toroidal magnet, m, and current, c. (a) Standard setup; (b) reciprocal setup; (c) topological invariance of the system.

The magnet need not be a torus *stricto sensu*; it can be a topological torus, idealized for example as a wire each line element of which carries a magnetic moment

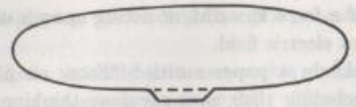

Fig. 2. Two contributions to the virtual work of the Biot-Savat force: one from the undefined part of the circuit; one from the closed circuit equivalent to the ideal one penetrated by the virtual deformation.

$\Phi d\mathbf{l}$, with Φ constant. Exhibiting no external poles, the magnet traps its flux Φ.

This interlaced system has the well-known mutual energy $W = I\Phi$ ($W = nI\Phi$ if there are n twists; $n = 0, \pm1, \pm2, \dots$); *none of it resides in the vacuum, as the magnet's field is identically zero.* Enclosed within the magnet and/or current, this energy W is topologically invariant: *if both loops are thought of as perfectly flexible and extensible, the system is in an indifferent equilibrium.*

Let us write down its equations in the simple case $n = +1$.

Expressed as residing in the current loop, the mutual energy

$$W = I\Phi \tag{14}$$

has the gauge-invariant form

$$W = I \oint \mathbf{A} \cdot d\mathbf{l}_c\,, \tag{15}$$

where $\mathbf{A}$ denotes the vector potential generated by the magnet. Expressed as residing in the magnet it has the form

$$W = \Phi \oint \mathbf{H} \cdot d\mathbf{l}_m\,, \tag{16}$$

where $\mathbf{H}$ denotes the magnetic field generated by the current, namely

$$\mathbf{H} = I \oint r^{-3}\mathbf{r} \times d\mathbf{l}_c\,. \tag{17}$$

Substituting this in (16) yields the far-action symmetric formula

$$WI\Phi \oint \oint r^{-3}\mathbf{r} \cdot [d\mathbf{l}_m \times d\mathbf{l}_c]\,, \tag{18}$$

which *must* be recovered by inserting in (15) the expression of the magnet's vector potential; *the Ampère gauge*

$$\mathbf{A} = \Phi \oint r^{-3}\mathbf{r} \times d\mathbf{l}_m \tag{19}$$

is thus uniquely selected.

Distributing the energy between magnet and current is of course not left to the choice of who writes the equations: according to Einstein's mass-energy equivalence, *W has physicality.* It can be shown[12] that there is equipartition.

Interestingly, a classical analog of the Aharonov-Bohm effect shows up via the emf $d\Phi/dt$ or $(dI/Idt)\Phi$ induced in the current loop by (respectively) varying Φ or I; the latter option is quite impressive, as no transient electric field appears.

Does this time variable emf have local, observable meaning? It certainly has in the former case, as $\partial_t\mathbf{A} = \mathbf{E}$ is an electric field; whether it has in the latter case, we leave as an open question.

6. Ampère Stress Tension $\mathbf{T} = I\mathbf{A}$ along a Current Loop

In a nutshell, cast in terms of the statics of (stiff in general) filaments, the far-action style magnetostatics of currents says[13]: *the stress tension* $\mathbf{T}$ *along a conducting wire of intensity* I *immersed in an external vector potential* $\mathbf{A}$ *is*

$$\mathbf{T} = I\mathbf{A}\,, \tag{20}$$

$\mathbf{A}$ *being expressed in the source-adhering gauge (3).*

This amounts to saying that each charge carrier is endowed with the "hidden momentum" QA previously discussed. No more compact expression of this stress tension in terms of the field magnitudes is available. As shown below, $\mathbf{A}$'s *gauge is selected as an integration condition via action-reaction opposition.*

It must of course be kept in mind that *a stress tension has potential meaning, being actualized only at severance points of a filament.* Also, as permanent currents exist only inside closed conducting wires, *the stress tension along a straight segment comes from curved adjacent parts.*

As $\mathbf{f} = d\mathbf{T}/d\mathbf{l}$ expresses the force density per line element, from (3) and the identity $\partial r^{-1} = -r^{-3}\mathbf{r}$ we derive the (widely used) formula of the directly opposite "shortened Ampère forces" exchanged between any two current elements:

$$d^2\mathbf{f}_A = -d^2\mathbf{f}'_A = II'r^{-3}(d\mathbf{l} \cdot d\mathbf{l}')\mathbf{r}\,. \tag{21}$$

Also, as $d\mathbf{C} = \mathbf{T} \times d\mathbf{l}$ expresses the torque applied to a line element, by substituting (3) we get the opposite torques exchanged between any two current elements

$$d^2\mathbf{C} = \pm II'r^{-1} d\mathbf{l} \times d\mathbf{l}'\,. \tag{22}$$

Direct linear and angular action-reaction opposition are evidenced in these formulas.

Conversely, assuming (21) and (22) as expressions of the mutual force and torque exchanged between any two current elements, that of the stress tension along a current carrying wire comes out as (20), with (3) included.

From this follows a natural representation of *directly opposing wrenches*[14] (combined force and torques: see Appendix, Section 19) exchanged between two complementary arcs PQ and QP of a current loop: *paired opposite tensions* $\pm\mathbf{T}_P$ *and* $\pm\mathbf{T}_Q$ *applied at the severance points* [calculated by integrating (21) and (22) over $\mathbf{dl}$ along PQ and over $\mathbf{dl}'$ along QP].

The Ampère gauge (3) is thus selected as an integration condition, via action-reaction opposition.

Compatibility of this Ampèrian scheme with the one based on the Biot-Savart (BS) force law has been extensively discussed[15] by the founding fathers of electrodynamics [the differential form of the BS formula is due to Laplace (1829) and to Grassmann (1845)].

For example, deforming at constant intensity I (with the help of an emf) a current loop immersed in a fixed vector potential $\mathbf{A}$, the varied mutual energy comes out as equal to the work of the BS force:

$$\delta W = I\delta\Phi = I\delta\oint \mathbf{A}\cdot\mathbf{dl} = I\iint \mathbf{B}\cdot[\mathbf{dl}\times\delta\mathbf{l}] = I\iint[\mathbf{B}\times\mathbf{dl}]\cdot\delta\mathbf{l}\,. \qquad (23)$$

Applied to the self-energy $(1/2)IO$ of a current loop, the same argument yields only *half* the work of the BS force: the other half must then go into a counter emf (cf. Fig. 2).

Momentum conservation is known to raise problems[16]: *directly opposing wrenches felt by complementary arcs PQ and QP of a circuit are not expressible in terms of the BS force* [while they obviously are in terms of the Ampère force (21) and torque (22)].

The BS, or Laplace-Grassmann, force applied to a current element $I\mathbf{dl}$ by another one $I\mathbf{dl}'$ is

$$d^2\mathbf{f}_L = I^2r^{-3}[\mathbf{dl}'\times\mathbf{r}]\times\mathbf{dl} \equiv d^2\mathbf{f}_A - I^2r^{-3}(\mathbf{dl}\cdot\mathbf{r})\mathbf{dl}'\,, \qquad (24)$$

$d^2\mathbf{f}_A$ denoting the (shortened) Ampère force (21). Thus *the nonzero sum* of the BS forces applied to each other by two current elements $I\mathbf{dl}$ and $I\mathbf{dl}'$ is

$$d^2(\mathbf{f}_L+\mathbf{f}'_L) = I^2r^{-3}[\mathbf{dl}'\times\mathbf{dl}]\times\mathbf{r}\,, \qquad (25)$$

so that *its double integral over* $\mathbf{dl}$ *along* PQ *and over* $\mathbf{dl}'$ *along* QP *is not identically zero.*

The traditional rejoinder starts from the remark that the last term in (24) cancels out in a closed integral over $\mathbf{dl}$, meaning that *the Ampère and BS forces applied to a current element by the rest of the circuit are equal.* But one contour integral, and a double integral over complementary arcs of a circuit, are quite different things.

It was then argued that a current's field is indivisible so that the partial field generated by a current element has no physicality. This hardly fits the Lorentz electron theory, where each electron in a line element $\mathbf{dl}$ acts upon each one in another one $\mathbf{dl}'$, *and does it according to the Lorentz formula.*

The action-reaction conundrum is thus transferred to the electrodynamics of point charges, where Wheeler and Feynman[17] have cleverly solved it (see Section 11).

The trick restoring formal action-reaction opposition in both Ampère's magnetostatic and Wheeler-Feynman's relativistic electrodynamics is: confer on the charge carriers the potential momentum $Q\mathbf{A}$ *or* QA^i, *the vector potential being the source-adhering Ampère or Liénard-Wiechert one; non-collinearity of velocity and momentum then follows; the* $I\mathbf{A}$ *tension and momentum* $Q\mathbf{A}$ *concepts clearly imply each other.*

To conclude: *Integrally equivalent to the Biot-Savart or Laplace-Grassmann scheme of magnetostatics which duplicates the Lorentz electrodynamics, there is an Ampèrian far-action scheme duplicating the Wheeler-Feynman relativistic electrodynamics. Linear and angular action-reaction opposition are evidenced in it.*

7. Self-Energy and Ampère Stress Tension

Denoting by k the (dimensionless) self-induction coefficient, by w the linear self-energy density, and by $\mathbf{t}$ the tangent unit vector along a current loop, the consequence $\mathbf{T}\cdot\mathbf{t} = (1/2)w = kI^2$ of (1), (20) and (21) exemplifies the usual proportionality between a trapped energy density and the pressure it exerts.

Due to its $1/r$ dependence, the self-vector potential $\mathbf{A}$ along a thin current is tangent to it. If then the ratio d/R of the wire's diameter to its curvature radius is everywhere small, the self-induction coefficient is constant. Thus $dA \equiv d(\mathbf{A}\cdot\mathbf{t}) \simeq 0$ and in an extension of the circuit, the varied self- energy very nearly equals half the work of the tension (as it does half the work of the BS force). Under this restriction an alternative expression of $dA \simeq 0$ is $\partial_u A_v dl^v \simeq 0$, entailing

$$f_u = \partial_v A_u dl^v \simeq [\partial_v A_u - \partial_u A_v]dl^v\,; \qquad (26)$$

the linear force density associated with the tension $\mathbf{T} = I\mathbf{A}$ *is equivalent to the Laplace-Grassmann one.*

The "linear pressure" T is displayed in Graneau's "exploding wires experiments" summarized in Section 9. In view of Saumont's experiments, also summarized, we notice that in a deformation affecting only an arc PQ of a circuit, two contributions to the varied self-energy can be considered: one due to the fixed part QP, and one equal to the self-energy of an ideal circuit made up of the difference between the final and initial positions of the arc PQ.

In the zero cross section limit k's expression diverges. Significantly, *flux quantization along a superconducting loop removes the divergence.* From the flux quantization formula

$$\Phi = \oint \mathbf{A} \cdot d\mathbf{l} = -\frac{nh}{2e} \quad (27)$$

and the expression of the current's intensity in terms of the frequency flow of electron pairs

$$I = -2e\nu \, , \quad (28)$$

we derive the self-energy quantization formula

$$2W = \Phi I = nh\nu \, . \quad (29)$$

8. Graneau's and Saumont's Experiments Evidencing the Vector Potential

Graneau's[18] "railgun" experiment is an enlarged version of Ampère's "hairpin" one. Feeling the BS force, a sliding bridge connecting two laterally constrained long straight parallel conductors generates by reaction a repulsive tension T along each wire, which can be evidenced at any abscissa x by inserting a mercury-loaded junction. If the emf generator is thought of as infinitely distant, T is x-independent, and unchanged in virtual displacements of the bridge; its relation to the virtual work is $2T = dW/dx$, like in a Swiss clock powered by a descending weight. According to Eq. (23) its expression IA is a corollary to that of the virtual work of the BS force applied to the bridge. Physicality of T then implies selection of a preferred gauge, the form of which can only be $A = kI$. *Existence of the Ampère tension is thus unquestionable.*

Two more sledgehammer proofs of existence of the repulsive tension $T = 2w$ ($w = dW/dl$) along a straight current are these: (1) the tangential tension T is radius-independent along a circular current, and (2) it is equivalent to the internal magnetic pressure $(1/4\pi)B^2$ stretching a long straight coaxial conductor capped by flat conducting discs; a and b denoting the inner and outer radii, the value $2w = I^2 \ln(b/a)$ is obtained either via $2w = I(A_a - A_b)$ with $A = I \ln r$, or by integrating $2\pi r B^2 dr$ from $r = a$ to $r = b$ with $B = I/2\pi r$.

Graneau's[18] "railgun" and "exploding wires" experiments I present by using his own words — with his permission and that of his publisher.[19] Un-indicated cuts are made, but of course the reader can go back to the original. My personal comments are inserted in square brackets.

While Graneau's evidencing of the repulsive Ampère tension has led him to adopt an anti-SRT stance, I deem the existence of this force not only quite compatible with relativistic covariance, but indeed *a corollary to the Lorentz force applied to the flowing electrons.*

Let us start with railguns (Ref. 19, pp. 156–8):

> "The gun consists of a pair of straight and parallel conductors one end [of which] is connected to a source of electric current. This end is the gun breech. The other end is the muzzle through which the projectile leaves. To begin with a short piece of copper bridges the rails near the breech. Called the armature [it is] in sliding contact with the rails. When a heavy current begins to flow down one rail, across the bridge, and back in the other rail, the armature is subjected to a strong electrodynamic force which accelerates it down the rails. This force is transverse to the current [in the bridge] and both Ampère's and Lorentz's laws agree on its magnitude. All guns are subject to recoil forces. The railgun recoil has been shrouded in mystery [with this I fully agree].
>
> Ampère's law claims that the forward force in the armature is balanced by two longitudinal rearward directed forces in the rail close behind the armature [which is quite well formalized by the $\mathbf{T} = I\mathbf{A}$ concept].
>
> [Thus] the rails will be pushed backwards, [deflected laterally, and buckled]. On the other hand, if Lorentz's law is correct the rails will not experience a recoil force and not buckle.
>
> [Therein lies a great shared misunderstanding: *along a straight tensed filament opposite tensions do exist:* a stretched string breaks under two repulsive tensions. To actualize *at any point*, and allow measurement of the tension dormant along a straight current, it suffices to insert a mercury loaded junction.]"

From railgun recoil and buckling we now turn to exploding wires (Ref. 19, pp. 148–151):

> "The response of fuse wires to large current pulses was studied in Warsaw by Jan Nasilowski. He noticed that a copper wire was shattered into small pieces by large but short current pulses. The wire showed no sign of melting. Metallurgical examination proved that every wire break was caused by an impulse tension. Not knowing of Ampère tension Nasilowski was baffled. [Long after] he was delighted with the Ampère force explanation."

If doubts remain concerning the existence of Ampère's repulsive stress tension along a straight current of axis z *and its rendering according to Maxwell's electromagnetism*, here is a rejoinder. Maxwell's "etheric" pressure $(1/8\pi)\mathbf{B}^2$ in the surrounding vacuum, integrated all over a plane orthogonal to z, builds up (as does any positive energy density) a repulsive tension just equal to $I\mathbf{A}$ [the logarithmic divergence in both can be removed by imagining a coaxial return current]. Assuming *adherence* between the field and its source explains the fact.

While Graneau plays in the large, Saumont[20] plays in the small.

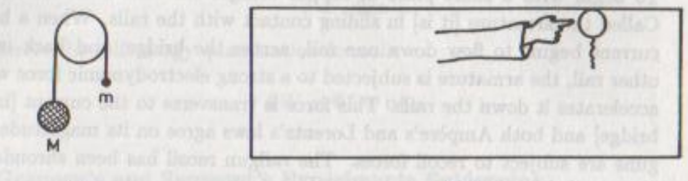

Fig. 3. (a) Atwood's machine: $\gamma = (M - m)/(M + m)$; (b) anti-recoil when pushing laterally a toy aerostat: the system's barycenter moves backward.

His horizontal armature (Fig. 3), moving vertically, is a straight wire some 13 cm long placed on the pan of a high precision mechanical (not electric) balance. Its ends, bent down wards, dip in mercury-loaded cups connected with an electric generator. The large fixed part of the circuit is horizontal and so, referring to formula (24) above, it contributes not to the virtual work — which then equals the self-energy of an ideal circuit defined as the difference between the final and initial positions of the armature.

Experimentation evidences, as expected, an upward lift proportional to the square intensity, of value independent of the armature's length.

The ends of the severed fix and mobile parts of the circuit are enameled, cut straight, with naked sections facing each other. Thus, *the force that is measured is parallel to the current inside the mercury, and repulsive.*

Counter-tests confirm this. If the armature's ends make U turns, so that the repulsive tension operates downwards, the measured force remains the same. If, inside the mercury, the current runs up at one end and down at the other, a zero force is measured. And if the current runs sideways in the mercury, a zero force is also measured.

Other counter-tests eliminate artefacts. Moved up or down, the fixed portion of the circuit regenerates the transverse BS force proportional to the armature's length. Care has been taken of the Archimedian lift on the armature's ends dipping in mercury, and of the aerostatic one due to Joule heating.

So illustrating the argument of Section 5, this set of experiments measures the sum and the torque of the mutually opposite wrenches applied to each other by complementary arcs of the circuit, in the form of paired opposite tensions of value $\pm I\mathbf{A}$ *locally tangent to the wire.*

J. Mourier,[21] expressing the self-induction coefficient k in terms of the ratio of diameter to curvature radius of the armature terminals, and using the local BS forces, gets good agreement with the measurements.

9. From Weber 1848 to Darwin 1920

Valid up to order $(v/c)^2$ included, Darwin's[22] 1920 Galilean-invariant, instantaneous-far-action approximation to electrodynamics is used in Coleman and van Vleck's[6] paper entitled "Hidden momentum in magnets." Let us evidence some far-reaching implications it has.

Defined *à la* Sommerfeld, the effective mass of each point charge is

$$m = m_0 + \frac{1}{2}c^{-2}(m_0v^2 + QV) + \dots , \tag{30}$$

V denoting the variable electric potential created by the other charges *in the Coulomb gauge.* The mutual energy *contribution* to the mass m has consequences we will discuss, We note first that *equipartition of the mutual energy of any two charges* is implied in formula (30).

Also follow[6] conservation of the system's total mass

$$M = \sum m , \tag{31}$$

barycentric moment

$$M\mathbf{R} = \sum m\mathbf{r} , \tag{32}$$

and total linear momentum

$$\mathbf{P} = M\mathbf{R}' = \sum m\mathbf{v} + \frac{1}{2}\sum\sum c^{-2}r^{-3}Q_aQ_br'^2\mathbf{r} + \dots . \tag{33}$$

Weber[24] found in 1848 the electric contribution to inertia and the direct action-reaction term displayed in formula (33); Darwin's more rigorous approach generates $Q\mathbf{A}$, not $QV\mathbf{v}$, in each particle's momentum, thus expressing non-collinearity of velocity and electromagnetic momentum; then it is via summation that $\sum Q\mathbf{A} \simeq \sum QV\mathbf{v}$. Darwin's expression of each particle's effective momentum is the low velocity approximation to Wheeler-Feynman's covariant formula (40) — which is especially relevant for us.

Helmholtz objected to Weber's formula that it allows negative masses, which is true of Darwin's formula also and is discussed in the next section: *a point charge immersed in a fieldless electric potential, the source of which is not freely falling, is not freely falling either.*

10. Coulomb Potential-Induced Archimedian Lift – or Rest

The Archimedian lift $-m\mathbf{g}$ felt by a body "displacing" a fluid mass m in the presence of a gravity field $\mathbf{g}$ mimicks anti-gravity. It also mimicks anti-inertia as, according to Einstein's inertia-gravity equivalence (including his elevator metaphor), a body immersed in a fluid, the container of which is accelerated by $\mathbf{g}$, displays a negative extra inertia $-m\mathbf{g}$. So *a body "displacing" a fluid mass m behaves as being endowed with a negative extra mass (not rest mass)* $-m$. For example, inside an accelerating (decelerating) caravan a toy helium balloon is projected forward (backward); and if inside a parked caravan someone displaces laterally a toy balloon resting at the ceiling via barycenter conservation the van anti-recoils, so that relative to the background the person is moved forward (cf. Fig. 3).

Darwin's formulas imply the existence of analogous electrodynamic phenomena. Consider, in place of the van, an insulating uniformly charged sphere with inside it a point charge replacing the balloon. In slow relative motion of the sphere, charge uniformity of the enclosed Coulomb potential *proper* is maintained by the sphere's variable stress tension. As the sum of the stress-energy plus the half-mutual-energy attached to the sphere remains zero, *totality of the electrostatic mutual energy VQ is reported on the point charge* — a conclusion already drawn in the Introduction from the timelike Aharonov-Bohm effect.

If both are at rest, sphere and charge exert no mutual forces. But if the sphere is accelerated from outside, or the charge from inside, formula (32) requires that the point charge of value Q should behave as being endowed with an extra mass $c^{-2}VQ$; Einstein's inertia-gravity equivalence then means that, inside a sphere at $V \geq +511\,\mathrm{kV}$, an electron will levitate and a firing electron gun will anti-recoil.

These outrageous claims are derived straightaway from Darwin's semi-relativistic electrodynamics, namely the Coleman-van-Vleck's barycenter formula (32); can we trust them?

Let us modify our thought experiment by placing the point charge Q on the axis z of a cylindrical capacitor, thus immersing it in a uniform potential of expression $V = q\ln(r_1/r_2)$. Pictured *à la* Maxwell, the mutual energy VQ resides in the field between the armatures with its barycenter at Q. Thus *if slowly accelerated, or held suspended in a uniform gravity field along z, the point charge, shadowed by this cloud, behaves as being endowed with the extra inertial mass* $c^{-2}VQ$.

11. Relativistic Far-Action-Reaction in the Wheeler-Feynman Electrodynamics

Explicitly covariant *à la* Minkowski, the Wheeler-Feynman[17] electrodynamics of point charges is best visualized as a four-dimensional transposition of the statics of filaments. An isomorphism exists between it and the Ampère magnetostatics discussed in Section 6, the translation lexicon being: 3-space → 4-space-time; current wire of intensity I → timelike trajectory of charge Q; stress tension → 4-momentum; linear force density → 4-force; torque → 6-angular momentum; energy → action.

Using Fokker's definition of the *mutual action* of two point charges ($i, j, k, l = 1, 2, 3, 4$; $x^4 = ict$; $r^i = a^i - b^i$)

$$d^2L_{ab} = Q_aQ_b\delta(r^2)da_idb^i \tag{34}$$

[compare with (2)] and the *source-adhering* half-retarded half-advanced Liénard-Weichert 4-potential [compare with (3)]

$$dA^i_{(a)} = \sum Q_b\delta(r^2)db^i\ , \tag{35}$$

which obeys the Lorentz condition

$$\partial_iA^i = 0\ , \tag{42}$$

WF derived for each point charge of combined 4-momentum

$$P^i = mV^i + QA^i \tag{36}$$

and therefore combined action

$$dL = P_idx^i\ , \tag{37}$$

the acceleration equation ($ds^2 = -dx_idx^i$)

$$dP^i = F^ids \tag{38}$$

evidencing *direct action-reaction* between any two point charges

$$d^2F^i = \pm Q_aQ_b\delta'(r^2)(da_jdb^j)r^i \tag{39}$$

[compare with (22)].

Integrally equivalent to this "Ampère 4-force," there is the *a and b asymmetric Lorentz 6-force* $Q_a B_{(a)}{}^{[ij]} da_j$ felt by each charge: from (35) we derive

$$Q_a dA^i(a) = \sum Q_b \delta'(r^2) r^i db^j \ ,$$

which, subtracted from (39), yields

$$dp^i{}_n = Q_a \sum Q_b \delta'(r^2)[r^j db^i - r^i db^j] da_j \ .$$

These WF formulas entail a largely overlooked important consequence: *any point charge immersed in a source-adhering 4-potential of expression (35) carries an extra inertia* QA^i.

For example, while the Coulomb interaction force between two heavy point charges at quasi-rest follows from (39), (36) confers on each an extra mass $(1/2)c^{-2}r^{-1}Q_aQ_b$. There also follows from (39) the Weber-Darwin velocity-dependent interaction force between moving charges — for instance, between two successive electrons in a cathodic beam.

As is well known,[25] non-collinearity of 4-velocity V^i ($V_iV^i = -c^2$) and 4-momentum entails existence of a 6-torque; here, a "potential" torque is defined:

$$C^{ij} = P^i V^j - P^j V^i \ , \tag{40}$$

the operational meaning of which is explained below. Opposite potential 6-torques

$$d^2C^{ij} = \pm Q_a Q_b (r^2)[da^i db^j - da^j db^i] \tag{41}$$

are applied to each other by any two interacting charges [compare with (22)].

It is highly significant that: (1) *use of the source-adhering Liénard-Wiechert gauge (35) is required for getting the direct action-reaction equation (39)*; (2) *from it there necessarily follows*[17] *the Lorentz condition*

$$\partial_i A^i = 0 \ . \tag{42}$$

From the first remark we derive a far-reaching, relativistically covariant statement: *accelerated by any means inside a given electromagnetic field, a point charge* Q *displays an extra inertial 4-momentum* QA^i, A^i *denoting the source-adhering 4-potential. This, together with Einstein's phenomenological inertia-gravity equivalent, validates our frequent recourse to ideally weighing a system (or parts of it) inside a uniform gravity field.*

We shall now discuss operationality of the "potential" 6-torque (41). Its three $[u4]$ "boost" components were met with in the discussion of equation (33); its three

We shall now discuss operationality of the "potential" 6-torque (41). Its three $[u4]$ "boost" components were met with in the discussion of equation (33); its three $[uv]$ components, angular momentum proper, display their magic in the following example.

Consider a charge Q flying at a velocity $\mathbf{v}$ parallel to a fixed infinitely long and thin straight magnet trapping a flux Φ. The paradox is that, while feeling no force, the moving charge exerts, via its magnetic field $\mathbf{B} = \mathbf{E} \times \mathbf{v}$, a torque on the magnet, the value of which is $\Phi[\mathbf{B} \times \mathbf{dl}] = -\Phi(\mathbf{v} \cdot \mathbf{dl})\mathbf{E}$ per line element. The integrated torque comes out as $-Q\Phi V/2r$, r denoting the constant distance between charge and magnet. As the magnet's fieldless vector potential is of the value $A = \Phi/2\pi r$, it turns out that *just opposite to the torque felt by the magnet there exists a "potential torque"* $Q[\mathbf{A} \times \mathbf{v}]$ *attached to the flying charge.*

Thus an angular momentum credit card (similar to the potential energy credit card so widely used) is here accepted. If the magnet of axis z is not infinitely long, but only very long, the charge, before and after running alongside it, passes near a pole, feeling there the Lorentz force generated by a magnetic field coplanar with z; there it initially deposits, and finally cashes, orbital angular momentum in the form discussed above.

The preferred gauge here is the one expressing the reaction from the potential's source.

12. Angular Action-Reaction Opposition, a Feynman Lectures Conundrum, and de Broglie's Photon Spin Density

Going back to Section 3, we replace in equation (11) the point charge by a uniformly charged circle coaxial with the dipole of moment $\mathbf{M}$ — a "parallel," in geographic jargon. Thus we are left with a purely angular momentum problem.

At the latitude α the circles's "potential angular momentum" has the value $VM \cos \alpha$. I will argue that: (1) de Broglie's[26] concept of an electromagnetic spin density of expression

$$\sigma = \frac{1}{4\pi}\{-\mathbf{A} \times \mathbf{E} + V\mathbf{H}\} \tag{43}$$

must enter angular momentum balance; (2) as a consequence, an *opposite angular recoil appears in the magnet.*

Readers of *The Feynman Lectures on Physics* are challenged[27] to refute so outrageous a claim, the hint being that not the magnet, but the field, is the place where the missing angular momentum builds up, in the form of orbital angular momentum of the "mutual" Poynting vector $\mathbf{E}_c \times \mathbf{H}_m$. Existence of a "bootstrap merry-go-round" then follows, as no photons are orbiting — and none could, because the sphere's energy quantum is the "anomalously" small one $e^2/R \simeq (2/137)h\nu$, with $\nu = c/R$.

Let us radicalize the problem[9] by placing, at the center of a uniformly charged insulating sphere, in its fieldless Coulomb potential $V = Q/R$, an initially unmagnetized dipole ferromagnet. If, magnetizing spontaneously later, it generates a magnetic moment $\mathbf{M}$ of axis z, the induced electric field confers on the sphere an angular momentum

$$\mathbf{C}_s = \frac{2}{3} V\mathbf{M} \tag{44}$$

($2/3 = 1 - 1/3$ expressing the difference between spin and precession mentioned in Section 3).

If M is positive, a positively charged sphere will thus start rotating eastward, like our Earth. As outside it the magnetic lines go from north to south, and as the electric field points out, the Poynting vector blows eastward, like the tradewind; *as its angular momentum has the same sign as the sphere's, it cannot compensate for it.*

Outside the sphere, at latitude a, the fields and potentials have as non-zero polar components

$$E_r = Qr^{-2}, \qquad B_r = 2Mr^{-3}\sin a, \qquad B_t = -Mr^{-3}\cos a\,,$$
$$V = Qr^{-1}, \qquad A_t = Mr^{-2}\sin a\,;$$

from these we get, outside the sphere, as density contributions to the Poynting vector's orbital angular momentum,

$$d^3C_P = \frac{1}{2}EB_t r\sin 2a = +\frac{1}{2}r^{-4}QM\sin 2a$$

and to de Broglie's spin,

$$d^3C_B = A_tE_r\cos a + V(B_r\cos a + B_t\sin a) = -r^{-4}QM\sin 2a\,.$$

whence

$$d^3(C_P + C_B)_{\text{out}} = -\frac{1}{2}r^{-4}QM\sin 2a\,.$$

Inside the sphere, where $\mathbf{E} \equiv 0$ and $V \equiv$ const, we get

$$d^3(C_B)_{\text{in}} = +\frac{1}{2}r^{-4}QM\sin 2a\,.$$

Integrating outside the sphere, from R to $+\infty$, we get

$$(\mathbf{C}_P + \mathbf{C}_B)_{\text{out}} = -\mathbf{C}_P = -\frac{2}{3}V\mathbf{M}, \tag{45}$$

a formula saying that *the (algebraic) sum of the Poynting vector orbital angular momentum and of de Broglie's spin density compensates exactly for the sphere's potential angular momentum* $\mathbf{C}_S$ Physicality of the electric potential and of the photon's spin density could hardly be evidenced better.

Inside the sphere, the only density contribution to the volume integrals is $V\mathbf{B} \equiv V\partial \times \mathbf{A}$, which ($V$ being constant) transforms into a surface integral of value

$$\mathbf{C}_{B\ \text{in}} = +\frac{2}{3}V\mathbf{M}\,, \tag{46}$$

just equal to the sphere's $\mathbf{C}_S$. We thus infer that *the magnet contains a "hidden angular momentum"* $-\mathbf{C}_S$ [note that $(2/3)V\mathbf{M}$ is the volume integral of $-V\mathbf{B}$].

Thus, *a zero total angular momentum state of the sphere-and-magnet system is such that both pieces contain opposite electromagnetic angular momenta — in accord with the action-reaction principle.*

A thought (or why not real?) test could use the Einstein-de-Haas (EdH) effect inside the sphere. At the voltage V such that the potential angular momentum $(2/3)V\mathbf{M}$ is just opposite to that $-(m/2e)\mathbf{M}$ due to the electron spin, the EdH effect should be inhibited. As, with $V_0 \simeq 511\,\text{kV}$, $(1/2)V_0\mathbf{M}$ expresses the ferromagnet's spin, this value is $V \simeq 383\,\text{kV}$ (beware: the *whole* experiment *must* be conducted at this voltage: uncharging the sphere before making the measurement would intercalate a counter effect).

To conclude: de Broglie's[26] concept of an electromagnetic spin density (cf. expression (43)): (1) restores the right sign in the Feynman Lectures[27] conundrum, (2) *validates action-reaction opposition between the sources of the (combined) field*, and (3) *confers a testable physicality on the electric potential.*

13. De Broglie's Photon Energy-Momentum and Spin Tensors

According to the theory of elasticity, $\mu^{uv} = E^{uv} - E^{vu}$ $(u, v = 1, 2, 3)$ is the local torque density; from this, one infers[25] that in any medium or field endowed with a spin density σ the energy-momentum density T^{ij} $(i, j, k, l = 1, 2, 3, 4;\ x^4 = ict)$ is asymmetric and obeys the relation

$$T^{kl} - T^{lk} = \partial_i\sigma^{i[kl]}\,, \tag{47}$$

where, inherently antisymmetric in its last two indexes, the spin density often is fully antisymmetric.

Using the standard Minkowski-Maxwell equations

$$\sum_{[ijk]} \partial^i B^{jk} = 0, \quad \partial_i H^{kl} = j^k, \quad B^{ij} = H^{ij} + M^{ij}\,, \tag{48}$$

$$B^{ij} = \partial^i A^j - \partial^j A^i\,, \tag{49}$$

we consider the following three asymmetric energy-momentum tensors: (1) Maxwell-Minkowski's

$$M^{ij} = -\frac{1}{4\pi}\left\{B^{ik}H^{j}{}_{k} + \frac{1}{4}B_{kl}H^{kl}\delta^{ij}\right\}, \tag{50}$$

(2) de Broglie's[26] photon canonical one, where $[\partial^i]$ denotes the Schrödinger or Gordon operator (difference between the partial derivative operators to the right and to the left)

$$P^{ij} = \frac{1}{4\pi}A_k[\partial^i]H^{jk}, \tag{51}$$

and (3) one implicit all through Sections 2, 4, 5, 7,

$$N^{kl} = -\frac{1}{4\pi}\left\{A^k j^l + \frac{1}{2}A_i j^i \delta^{kl}\right\}. \tag{52}$$

Using de Broglie's photon spin density

$$\sigma^{[ijk]} = \frac{1}{4\pi}\sum_{[ijk]} A^i H^{jk}, \tag{53}$$

and the magnetic polarization current density

$$l^{[ijk]} = \sum_{[ijk]} \partial^i H^{jk}, \tag{54}$$

we derive from the Minkowski-Maxwell equations (50) and (52) *plus the Lorentz condition (40)* the spin density conservation equation

$$\partial_i \sigma^{ijk} + A_i l^{ijk} = (M+N+P)^{jk} - (M+N+P)^{kj}, \tag{55}$$

to be compared with (48).

As is well known, the antisymmetric contribution to the Maxwell stress tensor contains the electromagnetic torque and boost densities, $\mathbf{E}\times\mathbf{P}+\mathbf{H}\times\mathbf{M}$ and $\mathbf{D}\times\mathbf{B}-\mathbf{E}\times\mathbf{H}$. Similarly, the antisymmetric contribution to the $A^k j^l$ tensor contains the torque and boost densities $\mathbf{A}\times\mathbf{j}$ and $q\mathbf{A}-V\mathbf{j}$ previously encountered.

* The 4-force densities attached to the three energy-momentum tensors are: (1) the sum of the Lorentz and the Curie or Stern-Gerlach ones,

$$\partial_k M^{ik} = B^{ik}j_k + \frac{1}{4}B_{kl}[\partial^i]H^{kl}; \tag{56}$$

(2) the sum of the Lorentz one, and of an unfamiliar one to be discussed in Section 15,

$$\partial_k N^{ik} = -B^{ik}j_k + \frac{1}{2}A^k[\partial^i]j_k; \tag{57}$$

* C.R.A.S. 263 1362 1966.

(3) twice the sum of the two Stern-Gerlach style ones,

$$\partial_k P^{ik} = -\frac{1}{2}B_{kl}[\partial^i]H^{kl} - A_k[\partial^i]j^k. \tag{58}$$

De Broglie's massive photon obeys Minkowski-Maxwell equations with Gordon's operator replacing d'Alembert's; Proca later proposed equivalent equations for a spin 1 meson.

Let us recall briefly how de Broglie's photon spin density shows up in a Maxwellian plane wave.

The general plane wave of time frequency v and space frequency $\mathbf{k}$ ($k=\nu/c$) can be thought of as a superposition of two waves of opposite circular polarizations. In a pure helicity state each photon has an energy hv and a spin $\pm h/2\pi$. The mutually orthogonal vectors $\mathbf{E}$ and $\mathbf{H}=2\pi\nu\mathbf{A}$, of the same magnitude in mixed units, rotate in the wave front, clockwise or anti-clockwise; $(1/2)(E^2+H^2)=nhv$ and $\mathbf{A}\times\mathbf{H}=\pm nh/2\pi$ respectively measure the energy and spin densities.

This holds for a massless photon. A massive photon flies not at the limiting velocity c, but tends to do so as its frequency increases. Spin zero momentum-energy carrying plane waves then exist, propagating a "longitudinal" electric field and vector potential; the probability of their excitation is exceedingly weak.[26]

14. Constant-Potential-Dependent Forces as Source Reactions

Associated with "hidden linear momentum in current loops" of expression (4) is the ponderomotive force $V(dI/dt)\mathbf{dl}$; and with the cemf generated along a varied current by a permanent flux from outside, there is the emf $I\mathbf{A}dI/dt$; and with "hidden angular momentum in magnets," there is the ponderomotive torque $Vd\mathbf{M}/dt$.

All these are reactions from the constant potential's source thought of as infinitely heavy.

15. On the $A^k j^l$ Energy-Momentum and $A^i m^{jk}$ Spin Tensors

The space-time scalar $A^k j_k$, a four-dimensional action density (and a three-dimensional energy density $Vq-\mathbf{A}\cdot\mathbf{j}$), is a familiar factotum in Lagrangians "where gauge dependence of just a calculation ingredient matters not."

The $q\mathbf{A}$ and $V\mathbf{j}$ components of the $A^k j^l$, tensor build up the respective "hidden momenta" in formulas (4) and (5).

The gauge-dependent 4-force density $A_k[\partial^i]j^k$ associated with this tensor in equation (58) is not unfamiliar, as its time component is the difference of known power densities; operationality of the space components then follows via relativistic invariance. Ponderomotive or electromotive force densities such as $V\partial_t\mathbf{j}$, $\mathbf{A}\partial_t q$, $\mathbf{A}\partial_t\mathbf{j}$, have repeatedly shown up in the preceding sections.

As said before, this 4-force density appears as relating the tension $\mathbf{T} = I\mathbf{A}$ and the Biot-Savat force density $I\mathbf{B} \times \mathbf{dl}$ along a current loop. In general we find $(u, v, w = 1, 2, 3)$

$$\frac{dT_u}{dl} \equiv \partial_w T_u dl^w = I[\partial_w A_v - \partial_v A_w]dl^w + I\partial_w A_u \cdot dl^w ; \tag{59}$$

that is, the sum of the BS force plus this force.

Interlaced with a toroidal magnet (see Section 4) a current loop feels no force. While the prevalent stance is "no Lorentz force, no force," our is: "no BS force, but exact compensation between tension and linear force density." *An analogous statement holds along the toroidal magnet of Section 4, where no one disputes the reality of the (unobserved) tension* $\mathbf{T} = \Phi\mathbf{H}$ *and of the Stern-Gerlach force* $f_u = \Phi\partial_w H_u \cdot dl^w$.

The spin density $V\mathbf{m}$, evidenced in formulas (13) and (46), is a component of the tensor $A^i m^{[jk]}$.

So the $A^k j^l$ and $A^i m^{jk}$ tensors are implied in any covariant formulation of the phenomena discussed in this essay.

16. Lorentz Condition Revisited

The Lorentz condition, a very strong restriction on the gauge, requires that the arbitrary scalar superpotential be a solution of d'Alembert's equation — that is, a sourceless propagating field magnitude.

The following statement then holds: *the general solution of Maxwell's equations is the sum of physical source-adhering solutions plus the ghostlike general solution of the homogeneous d'Alembert equation.*

Plane wave d'Alembert solutions are "longitudinal waves" propagating a fieldless lightlike 4-potential, Discarding them as "unphysical" is *a contrario* conceding some physicality to the transverse potential ...

Finally, how does the Lorentz condition $\partial_k A^k = 0$ dovetail with the source-adhering gauges?

Consider permanent regime cases, such as electrons orbiting a heavy nucleus, or Aharonov-Bohm ones bypassing a long solenoid. A permanent regime is one such that in some inertial frame all time derivatives of the field magnitudes are zero; so $\partial_t V = 0$ and, via the Lorentz condition, $\partial \cdot \mathbf{A} = 0$ Thus the central Coulomb gauge compatible with the Lorentz condition is defined as $V = Q/r$ plus $\mathbf{A} = 0$; and the cylindrical Ampère gauge compatible with the Lorentz condition as $\mathbf{A} = r^{-2}\Phi \times r$ plus $V = 0$ (Φ denoting the magnetic moment per line element). In either case, we require that the 4-vector A^i be zero at spatial infinity. The Liénard-Wiechert gauge (retarded, advanced, or time-symmetric) — a four-dimensional extension of the Coulomb and Ampère static gauges — satisfies the Lorentz condition.

So *the Lorentz condition is compatible with any source-adhering gauge defined covariantly.*

As source-adhering solutions of Maxwell's equations are the zero mass limits of massive photon solutions, their use should be the norm.

17. Lorentz Condition in the Dirac Electron Theory

All textbooks explain gauge invariance of the Dirac electron theory, none mentioning, however, that *the Lorentz condition is inherent* — as otherwise a scalar field $\partial_i A^i$ would come up in the second order equation. And none mentions either the following *corollary.*

In 4-frequency terms, any gauge field obeying d'Alembert's equation $k_i k^i = 0$ and the Lorentz condition $k_i A^i = 0$ constrains the gauge to $A_i A^i = 0$. *Thus a gauge change on the free electron amounts to adding a lightlike vector to its 4-frequency: an extreme-relativistic Lorentz transform implying zero electron mass.* The conclusion then is: *in the unbounded vacuum the electron's gauge must be* $A^i \equiv 0$.

Consequently, a source-adhering gauge should be selected for any electromagnetic potential embedding the electron — including fieldless ones. *Such a gauge is selected not from within, but from without the Dirac equation.*

This is how action-reaction with an infinitely heavy source is taken care of: a "macroscopic renormalization" of mass (not rest mass), momentum, or angular momentum. For example, the zero of the energy eigenvalue of an electron guided along the axis of an infinitely long cylindrical capacitor, thus bound to a positive potential V, is at $-c^{-2}eV$. The same is true for a hydrogen atom placed at the center of a large uniformly charged sphere. Of course, part of the electronic wave tunnels outside the container.

Anyone familiar with the Dirac equation knows existence of the $10 = 2 \times 5$ tensorial equations obtained by adding and subtracting the Dirac equation and its adjoint after multiplying the former to the left by $\bar{\Psi}\gamma$ and the latter from the right by $\gamma\Psi$; these are known[28] as the Franz-Kofink equations. In the absence of a 4-potential this system consists of two uncoupled subsystem of five equations, one of electromagnetic and one of mechanical meaning. A non-zero A^i thus generates an electro-mechanic coupling that is *symmetric* in the following sense.

Ten gauge-invariant space-time tensors exist in Dirac's theory, namely: five Dirac style $\bar{\Psi}\gamma\Psi$ and five Gordon or Schrödinger style $\bar{\Psi}\{(ih/4\pi)[\partial^i] + eA^i\}\Psi$, $[\partial^i]$ denoting the Gordon operator, the difference between the partial derivative operators to the right and to the left; *both contributions in the latter five tensors are gauge-dependent.* Five of *either* the Dirac *or* the Schrödinger ten tensors have an electric and five a mechanic meaning.

The *symmetric, or crossed, coupling mediated by* A^i consists of this: the Dirac style "potential" contribution in respectively an electric (mechanic) Gordon tensor is mechanic (electric). *Thus A^i behaves as an electro-mechanic potential.*

There are three mechanical Schrödinger style tensors, all of interest to us: the canonical rank 2 energy-momentum density T^{kl}, its trace T, and the canonical rank 3 spin density $\sigma^{i[jk]}$. The potential contributions in these, namely $A^k j^l$, $A_k j^k$, and $A^i m^{jk}$ (j^k and m^{jk} denoting Dirac's 4-electric current and 6-polarization densities), have repeatedly shown up in the preceding sections: $A^k j^l$ contains the "hidden momentum densities" $V\mathbf{j}$ and $q\mathbf{A}$ and, by contraction, the energy density $\mathbf{A}\cdot\mathbf{j}$; $A^i m^{jk}$ contains the "hidden angular momentum density" $V\mathbf{m}$.

So, *of the two gauge-dependent contributions in these gauge-invariant tensors, the "potential" one expresses the ponderomotive force or torque, the other one the electromotive force or torque.*

18. Conclusion: Electro-gravific Interaction

That contact-action and far-action are alternative conceptualizations and formalizations of electrodynamics has been exemplified *à la* Diogenes by Wheeler and Feynman.[25] In an earlier paper[29] they had shown mathematical equivalence of Maxwellian retarded solutions including the Lorentz damping force, or advanced solutions including a Lorentz anti-damping force, or half-retarded half-advanced solutions with no damping force *insofar as nothing other than mechanical inertia and electromagnetic interaction is considered.*

Electro-gravific interaction, a leitmotiv throughout our essay, includes the statement that *QA^i with A^i expressed in the source-adhering gauge is an extra inertia to a charge immersed in a given electromagnetic field* — one more hint that the 4-potential is a keystone between electricity and gravity. If so, the "de Broglie formula"

$$P^i = m_0 V^i - eA^i = \frac{h}{2\pi} k^i \tag{61}$$

should be contemplated in its full-fledged expression[30] rather than in the shortened form generally quoted.

The presently accepted "Einstein-Maxwell theory" inserts the symmetric free field Maxwell stress tensor in the source of gravity. Our arguments, all expressed in terms "manifestly compatible" with the SRT, converge towards the idea that *the symmetric $A^k j^l$ tensor should be included as gravity source in an "Einstein-Cartan"[31] formalism.* Explanation of effects such as Graneau's and Saumont's, and prediction of others like linear and angular recoil effects, have been proposed. Via asymmetry of the energy-momentum tensors, spin, together with boost its twin, enter the picture — which automatically follows if interacting Dirac electrons and de Broglie massive photons are used as gravity sources.

19. Appendix: Euclidean-Galilean Rigid Bodies and Wrenches

The general Euclidean displacement of a rigid body consists of a translation and a rotation. At any Galilean time t a rigid body moving in an inertial frame is thus endowed with "instantaneous" linear $\mathbf{V}$ and angular $\mathbf{A}$ velocities. Between the velocities $\mathbf{P}'_i$ and $\mathbf{P}'_j$ of two points $\mathbf{P}_i$ and $\mathbf{P}_j$ in the solid, there exist the vector relations

$$\frac{1}{2}\{(\mathbf{P}_i - \mathbf{P}_j)^2\}' = (\mathbf{P}_i - \mathbf{P}_j)\cdot(\mathbf{P}'_i - \mathbf{P}'_j) = 0\,,$$
$$\mathbf{P}'_j - \mathbf{P}'_i = \mathbf{A}\times(\mathbf{P}_j - \mathbf{P}_i)\,.$$

The condition $\mathbf{A}\times\mathbf{P}'_i = 0$, $\mathbf{A}\neq 0$, uniquely defines the axis of the A&V wrench and the "tangent helical motion."

The wrench concept also turns up in the statics of solids. A system of forces applied at points of a solid has a "sum" $\mathbf{S}$ and, at any point $\mathbf{P}_i$, a "moment" $\mathbf{M}_i$; the following vector relations hold:

$$(\mathbf{P}_i - \mathbf{P}_j)\cdot(\mathbf{M}_i - \mathbf{M}_j) = 0\,,$$
$$\mathbf{M}_j - \mathbf{M}_i = \mathbf{S}\times(\mathbf{P}_j - \mathbf{P}_i)\,.$$

Again, if $\mathbf{S}\neq 0$, there is an "axis" along which $\mathbf{S}\times\mathbf{M}_i = 0$.

Not only in the statics and kinematics, but also in the dynamics of solids, is the wrench concept useful: the power exerted by a wrench of forces moving a solid is $\mathbf{S}\cdot\mathbf{V}+\mathbf{M}\cdot\mathbf{A}$.

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THE SYMMETRY BETWEEN ELECTRICITY AND MAGNETISM AND THE PROBLEM OF THE EXISTENCE OF A MAGNETIC MONOPOLE

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1. Introduction

It seems fair to say that there are about as many physicists who consider the magnetic monopole as a monster hidden in the depths of the Loch Ness, as there are who regard this idea as so necessary for the beauty of nature, that God cannot possibly have failed to think about it. I belong of course to the latter species !

It is well known that the hypothesis of separated magnetic poles is an old one, but the present paper is neither devoted to its history nor to a comprehensive bibliography on the subject : there already exist several papers or books of this kind [1], [2], [3], [4]. Here, we shall quote only those papers that are useful for our purpose, which is to give arguments in favor of the hypothesis of magnetic monopoles, the possibility of their observation and the explanation of the fact that they were not yet observed with certainty.

Therefore, we shall not survey all aspects of the problem. In particular, although this is a commonly favoured point of view, there will be no further mention of a possible hyper-heavy monopole. Keeping away from G.U.T., we shall remain in the framework of electrodynamics. On the other hand, we shall not confine ourselves to symmetry arguments, but shall present a *wave equation* for a magnetic monopole, which parallels the Dirac equation for the electron. This equation describes a monopole quite different from the one which is usually considered, but it satisfies all the electrodynamical, mechanical and gauge properties